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Conley Index Approach to Sampled Dynamics[*]

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Abstract. The topological method for the reconstruction of dynamics from time series [K. Mischaikow et al., *Phys. Rev. Lett.*, 82 (1999), pp. 1144–1147] is reshaped to improve its range of applicability, particularly in the presence of sparse data and strong expansion. The improvement is based on a multivalued map representation of the data. However, unlike the previous approach, it is not required that the representation has a continuous selector. Instead of a selector, a recently developed new version of Conley index theory for multivalued maps [B. Batko, *SIAM J. Appl. Dyn. Syst.*, 16 (2017), pp. 1587–1617; B. Batko and M. Mrozek, *SIAM J. Appl. Dyn. Syst.*, 15 (2016), pp. 1143–1162] is used in computations. The existence of a continuous, single valued generator of the relevant dynamics is guaranteed in the vicinity of the graph of the multivalued map constructed from data. Some numerical examples based on time series derived from the iteration of Hénon-type maps are presented.

Key words. nonlinear dynamics, chaos, topological semiconjugacy, topological data analysis, dynamical system, Conley index, periodic orbit, fixed point, invariant set, isolating neighborhood, index pair, weak index pair

AMS subject classifications. 54H20, 37B30, 37M05, 37M10, 54C60, 37B35

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1. Introduction.

Conceptual models for most physical systems are based on a continuum; values of the states of a system are assumed to be real numbers. At the same time science is increasingly becoming data driven and thus based on finite information. This suggests the need for tools that seamlessly and systematically provide information about continuous structures from finite data and accounts for the rapid rise in use of methods from topological data analysis (TDA). However, not surprisingly, there are significant challenges associated with the sampling or generation of data versus the necessary coverage from which to draw the appropriate conclusions. In this paper we focus on this challenge in the context of nonlinear dynamics.

The fundamental work of Niyogi, Smale, and Weinberger [29] provides probabilistic guarantees that the correct homology groups have been computed, but is based on uniform sampling of the manifold. For a nonlinear dynamical system one expects that the sampling is

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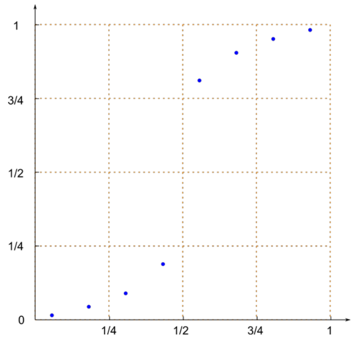
influenced by an underlying invariant measure that is rarely uniform with respect to the volume of the underlying phase space. Furthermore, in practice one seldom knows the underlying subset of phase space on which the dynamics of interest occurs, e.g., the invariant set. As a consequence one must expect that in applications we will need to collect considerably more data than a theoretical minimum would necessitate.

The predominant tool used by the TDA community to overcome the problem of lack of knowledge of the topological space of interest is persistent homology that provides homological information at all scales. There are two challenges associated with this approach. The first is that persistent homology computations on large data sets can be prohibitively expensive (there is extensive work being done to address this problem [9, 30, 17]) and, second, that the development of a persistence theory of maps is in its early stages [10, 11, 4]. An alternative technique is to bin the data. This is the approach we adopt in this paper. In particular, we assume that the data points are measured via coordinates and thus the binning in phase space naturally takes the form of cubical sets. The advantage is that we can a priori choose the bins so that the homological computations are feasible given time and memory constraints, and almost tautologically the binning process is a data reduction technique.

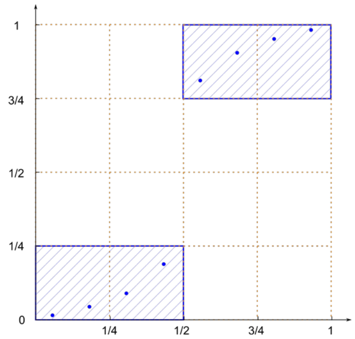
Identification of the space is only part of the challenge of understanding dynamics; we also need to capture the behavior of the nonlinear map that generates the dynamics. Though an oversimplification, interesting dynamics is often driven by nonlinearities that exhibit significant expansion. As is made explicit in [12] the amount of data needed to expect a correct direct computation of the induced maps on homology is proportional to the magnitude of the Lipschitz constant of the map. This will not be a surprise to anyone who has attempted to construct explicit simplicial maps for nonlinear functions. The significance of the work reported in this paper is that we can obtain reliable information about the dynamics without directly identifying the map.

To explain the philosophy before becoming submerged in the technical details (precise definitions and notation are provided in the following sections), consider a dynamical system on the unit interval and assume that we have collected the data $\{(x, y) \in [0, 1] \times [0, 1]\}$ as indicated in Figure 1.1(a). We interpret these data as providing information about the graph of a continuous map $f : [0, 1] \to [0, 1]$ and the question we ask is, *can we extract information about the dynamics generated by* f? The answer is yes. In fact, under minimal hypotheses we can conclude that there are attractors that contain a fixed point within the intervals $[0, \frac{1}{4}]$ and $[\frac{3}{4}, 1]$, and there exists an unstable invariant set, also containing a fixed point, in the interval $[\frac{3}{8}, \frac{5}{8}]$. These results are obtained by building an upper semicontinuous acyclic multivalued map $F : [0, 1] \rightrightarrows [0, 1]$ (see section 2) from the available data, applying to it a recently developed new version of Conley index theory for multivalued maps [2, 1] in order to identify isolating neighborhoods and index pairs, and then computing the associated Conley indices (see Definition 2.2). The last point requires that we be able to compute an induced map on homology.

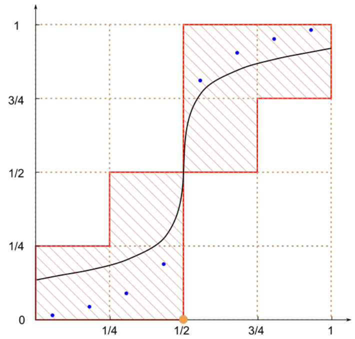
An outline for the strategy used to perform these identifications is as follows. As indicated above we bin the data. Using intervals of length 1/4 to define the bins we obtain the blue shaded regions shown in Figure 1.1(b). The blue regions are meant to provide a representation F of the graph of the unknown function f. Of course, as presented this is impossible; the domain of F is connected but the blue regions are not. One means of addressing this issue is



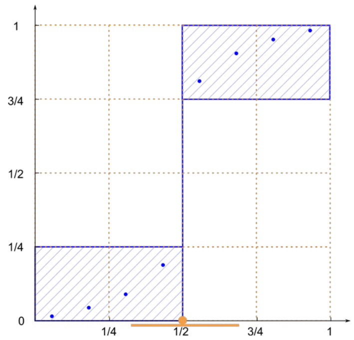
(a) The data marked by blue dots and the grid indicated with orange dashed lines.



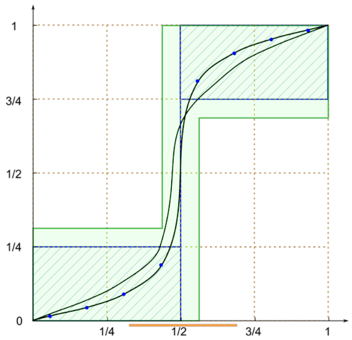
(b) The bins of data indicated with four squares shaded with blue.



(c) The expansion of bins indicated with six squares shaded with red and the graph of a continuous selector in black.



(d) The graph of an upper semicontinuous acyclic map F : X (X in blue. Isolating neighborhood N marked by orange line segment.



(e) The graph of map $F : X$ (X in blue, and its vicinity for continuous maps sharing with F isolating neighborhood N and the Conley index, in green.

Figure 1.1. *Construction of an upper semicontinuous acyclic multivalued map F covering points representing the data.*

to expand the representation so that the graph of a continuous function can be included in the representation, i.e., the representation admits a *continuous selector*. Techniques of this type were successfully employed in [23]. However, they may easily fail. Applying the method of [23] to the representation in Figure 1.1(b) leads to the representation in Figure 1.1(c). Actually, this is a minimal expansion which admits continuous selectors satisfying $f(\frac{1}{2}) = \frac{1}{2}$. However, the resulting approximation of the dynamics is too crude: the combinatorial procedure for finding isolation neighborhoods presented in [36, 37] fails to produce an isolating neighborhood for the fixed point $x = \frac{1}{2}$. On one hand, one can easily check that any other procedure must fail in this case, because the identity map is among selectors. On the other hand, using an even larger expansion that produces an outer approximation [20] and using methods detailed in [7, 8], the desired isolating neighborhood and index pair can be recovered. However, our experience is that applying this latter approach to complex time series data even for 2-dimensional examples, often results in failure.

Lest the reader think that this is a contrived example, consider the function $f : [0, 1] \to [0, 1]$ given by $f(x) = -nx^3 + (1 + n)x$ and observe that for $n \geq 1$ the points in Figure 1.1 are consistent with data lying near the graph of f. The dynamics generated by f consists of stable fixed points at 0 and 1, an unstable fixed point at 1/2, and connecting orbits from the unstable fixed point to the stable fixed points. Furthermore, as n increases, the minimal Lipschitz constant of f given by $f'(1/2) = \frac{4+n}{4}$ increases which results in the dynamics becoming more pronounced. However, from the perspective of experimental or numerically derived data, we expect the data points to cluster along the lines $y = 0$ and $y = 1$, and thus the observed discontinuity becomes more pronounced especially if one refines the binning. We take this to be yet another suggestion that the direct approach of constructing a representation that admits a continuous selector is not the ideal technique.

As indicated above, we draw conclusions about the continuous dynamics from induced maps on homology via the Conley index. This suggests that to obtain motivation for an alternative approach we consider the example from a purely homological perspective. Consider a function $f : [0, 1] \to [0, 1]$ and its graph $G_f := \{ (x, y) \in [0, 1]^2 \mid y = f(x) \}$. Let $\pi_1 : G_f \to [0, 1]$ and $\pi_2 : G_f \to [0, 1]$ denote the projections from the graph to the domain and range of f, respectively. Then π_1 is a homeomorphism, π_{1*} is invertible, and, on the level of homology, $f_* = \pi_{2*} \circ \pi_{1*}^{-1}$. Observe that if we replace G_f by the blue shaded regions shown in Figure 1.1(b) then π_{1*} is not invertible, but we still can deduce the correct map induced by F on homology. This is because the preimage π_{1*}^{-1} takes on two values, but these values are mapped to the same value under π_{2*}. For a more complete discussion on this perspective see [16]. What should be clear is that to apply this in general we require a condition that forces π_{2*} to collapse appropriate generators in the homology of the representation $H_*(F)$.

With this in mind consider the blue region shown in Figure 1.1(d). In this case the fibers of π_1 are acyclic, thus π_{1*} is invertible, and the question of how π_{2*} acts on generators is resolved. Because we are interested in extracting dynamics, rather than considering the blue region to be a fiber bundle over the phase space, we view it as the graph of an upper semicontinuous acyclic multivalued map $F : [0, 1] \rightrightarrows [0, 1]$ and we use F to extract isolating neighborhoods, index pairs, and, ultimately, the Conley index.

We note that in this simple 1-dimensional example, the choice of the blue line in Figure 1.1(d) is obvious. In higher dimensions there are a variety of means of attempting to resolve the issue of controlling how π_{2*} acts on generators from the preimage of π_{1*} and the identification of optimal methods remains an open question. In this paper we seek minimal rectangular regions.

To be more specific we assume that our data consist of a finite set of points $A \subset \mathbb{R}^d$ and our understanding of the dynamics is to be derived from the map $g : A \to \mathbb{R}^d$. We also assume that we have chosen a scale $\delta > 0$ for the binning and that the bins take the form

$$[n_1\delta, (n_1 + 1)\delta] \times [n_2\delta, (n_2 + 1)\delta] \times \cdots \times [n_d\delta, (n_d + 1)\delta],$$

where $n_i \in \mathbb{Z}$. More generally, we work with *δ-cuboids*, sets of the form

$$[n_1\delta, m_1\delta] \times [n_2\delta, m_2\delta] \times \cdots \times [n_d\delta, m_d\delta], \tag{1}$$

where $(n_1, n_2, \ldots, n_d), (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$. An *elementary cube* is a cuboid where $m_i - n_i \in$

$\{0,1\}$ for $i = 1, 2, \ldots, d$. We denote the set of all δ-cuboids in $\mathbb{R}^d$ by $\mathcal{C}^d_\delta$ and the set of all elementary δ-cubes in $\mathbb{R}^d$ by $\mathcal{K}^d_\delta$.

For a bounded subset $X \subset \mathbb{R}^d$ we introduce the following notation:

$$\mathcal{K}_\delta(X) := \bigcup \{ Q \in \mathcal{K}^d_\delta \mid X \cap Q \neq \emptyset \}$$

and

$$\lceil X \rceil_\delta := \bigcup \{ Q \in \mathcal{K}^d_\delta \mid \operatorname{conv}(X) \cap Q \neq \emptyset \},$$

where $\operatorname{conv}(X)$ denotes the convex hull of X.

Returning to the map $g : A \to \mathbb{R}^d$ its *sunflower enclosure* is the multivalued map $F^s_{g,\delta} : \mathcal{K}_\delta(A) \rightrightarrows \mathbb{R}^d$ defined by

$$F^s_{g,\delta}(x) := \lceil g(\mathcal{K}_\delta(x) \cap A) \rceil_\delta \subset \mathbb{R}^d.$$

We note that the map has nonempty values, because for $x \in \mathcal{K}_\delta(A)$ the set $\mathcal{K}_\delta(x) \cap A \neq \emptyset$ even if $x \notin A$. We leave it to the reader to check that given $\{ (x, g(x)) \in [0,1] \times [0,1] \}$ as shown in Figure 1.1(a), the graph of $F^s_{g,\delta}$ is as shown in Figure 1.1(d).

Sunflower enclosures satisfy a variety of nice properties. Recall (cf. [18]) that $F : X \rightrightarrows \mathbb{R}^d$ is *cubical* if

(a) $X \subset \mathbb{R}^n$ is a cubical set, i.e., it can be written as a finite union of elementary cubes;
(b) for any $x \in X$ the set $F(x)$ is cubical;
(c) for any elementary cube $Q = [n_1\delta, m_1\delta] \times \cdots \times [n_d\delta, m_d\delta]$ in X, $F_{|\mathring{Q}}$ is constant, where $\mathring{Q} := (n_1\delta, m_1\delta) \times \cdots \times (n_d\delta, m_d\delta)$ and $(n_i\delta, m_i\delta) = \{n_i\}$ if $n_i = m_i$.

The following proposition follows from [14, Proposition 14.5].

Proposition 1.1. *A sunflower enclosure is an upper semicontinuous cubical map.*

When the values of the sunflower enclosure are contractible, then using algorithms developed in [36] and the formula from [1, Theorem 4.4] one can identify cubical isolating blocks, cubical weak index pairs, and an index map associated with $F^s_{g,\delta}$ (see [31] for more details). In particular, a Conley index can be computed.

From the perspective of identifying dynamics the aforementioned computation should be viewed as purely formal, e.g., in and of itself it does not guarantee that there is a continuous map that generates dynamics that is compatible with the associated Conley indices. The majority of this paper is dedicated to guaranteeing that the formal computation does in fact lead to the existence of a large, but explicit, family of nonlinearities that are capable of producing the observed dynamics. To state our goals more precisely we introduce the following notation. Let $F : X \rightrightarrows X$. For simplicity of notation we identify F with its graph $\{ (x, y) \in X \times X \mid y \in F(x) \}$. Using the max-norm on the product space $X \times X$, let $B(F, \varepsilon) \subset X \times X$ denote the open set of points within ε of the graph of F (see Figure 1.1(e)). Following [15] (cf., e.g., [14]) we say that a continuous single valued map $f : X \to X$ is a *continuous ε-approximation (on the graph) of* $F : X \rightrightarrows X$ if $f \subset B(F, \varepsilon)$.

We denote the set of continuous ε-approximations of F by $a_\varepsilon(F)$.

Our claim is that Conley index information computed for $F : X \rightrightarrows X$, an acyclic upper semicontinous cubical map, is valid for the dynamics generated by any continuous function

$f \in a_\varepsilon(F)$ for all $\varepsilon \in (0, \varepsilon_0)$ sufficiently small. As the results described below indicate, our approach provides explicit lower bounds on ε_0.

We have, up to this point in the introduction, been rather circumspect about how the Conley index provides information about nonlinear dynamics. One of the more powerful results is that it can be used to construct semiconjugacies to known dynamics. To be more precise, given two continuous maps $f : X \to X$ and $\sigma : Y \to Y$, f is *semiconjugate* to σ if there exists a continuous surjective map $\rho : X \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho"'] & X \arrow[d, "\rho"] \\ Y \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

commutes. Semiconjugacies are of interest if the dynamics of σ is understood, as this implies that the dynamics of f must be at least as complicated, i.e., one can deduce structure about the dynamics of f from that of σ.

In the context of the Conley theory, one begins with an index pair $P = (P_1, P_2)$ (see section 2 for precise definitions). The homological Conley index is derived from a map $f_{P*} : H_*(P_1/P_2, [P_2]) \to H_*(P_1/P_2, [P_2])$ that itself is derived from the action of f on the pointed quotient space $(P_1/P_2, [P_2])$. Let $N = \mathrm{cl}(P_1 \setminus P_2)$. The meta form of the desired theorem is that given the homological Conley index, information about the index pair, and an explicit dynamical system $\sigma : Y \to Y$, then there exists a semiconjugacy

$$\begin{tikzcd} \mathrm{Inv}(N, f) \arrow[r, "f"] \arrow[d, "\rho"'] & \mathrm{Inv}(N, f) \arrow[d, "\rho"] \\ Y \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

where $\mathrm{Inv}(N, f)$ denotes the maximal invariant set in N under f.

The potential of the proposed theory in applications is demonstrated in [3], in particular in examples based on the time series studied in [23]In this paper we will prove the following three results.

Theorem 1.2. *Consider the time series* $\bar{x} = (x_i)_{i=100}^{20689}$ *generated by iterating the Hénon map*

$$H : \mathbb{R}^2 \ni (x, y) \mapsto (1 - ax^2 + by, x) \in \mathbb{R}^2$$

with the parameter values $a = 1.65$, $b = 0.1$, *and initial condition* $(x_0, y_0) = (0, 0)$. *Set*

$$A_{\bar{x}} := \{ (x_i, x_{i+1}) \mid i = 100, \dots, 20{,}688 \}$$

and let $g_{\bar{x}} : A_{\bar{x}} \to \mathbb{R}^2$ *be given by* $g_{\bar{x}}(x_i, x_{i+1}) = (x_{i+1}, x_{i+2})$.

Choose a binning of $\mathbb{R}^2$ *based on* $\delta := 0.008127$ *and let* $F := F^s_{g_{\bar{x}},\delta} : K_\delta(A_{\bar{x}}) \rightrightarrows \mathbb{R}^2$ *be the sunflower enclosure of* $g_{\bar{x}}$, *i.e.,*

$$F^s_{g_{\bar{x}},\delta}(x) := \lfloor g_{\bar{x}}(K_\delta(x) \cap A_{\bar{x}}) \rfloor_\delta \subset \mathbb{R}^2.$$

Let $\varepsilon = \delta/2$.

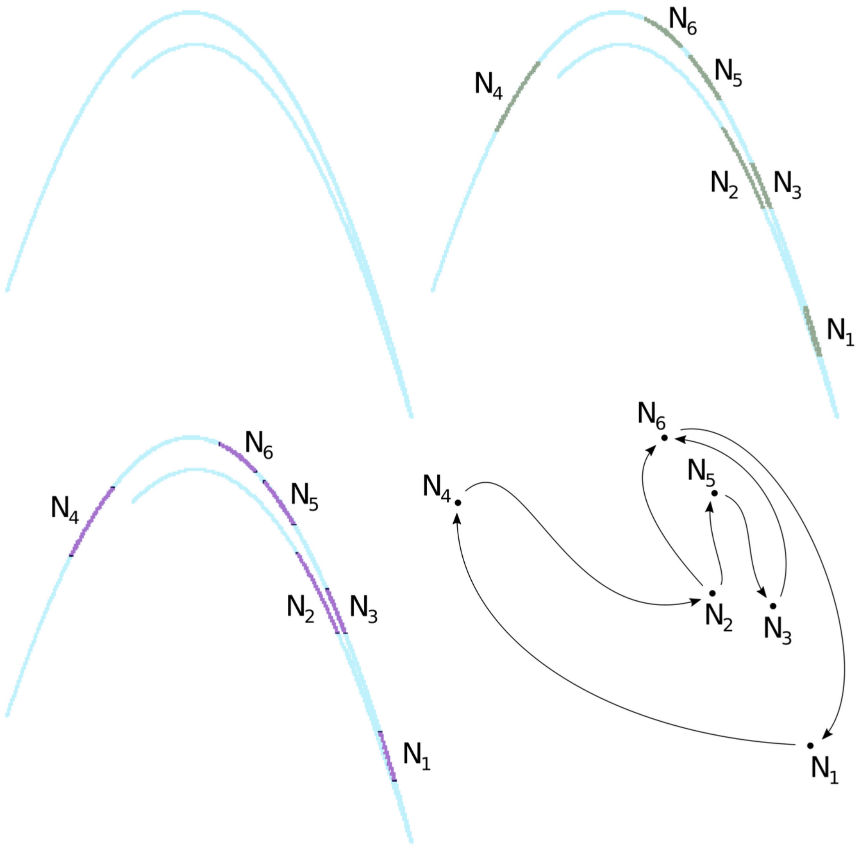


Figure 1.2. *Domain of sunflower enclosure for $g_{\bar{x}}$ consisting of 1184 2-dimensional cubes, an isolating neighborhood (in dark sea green), its weak index pair (in blue violet), and the graph of transitions between components of an isolating neighborhood.*

Then, $a_\varepsilon(F) \neq \emptyset$. Furthermore, there exists a compact set $N \subset \mathbb{R}^2$ (see Figure 1.2) such that for any $f \in a_\varepsilon(F)$

(i) *N is an isolating neighborhood of f;*

(ii) *there exists a semiconjugacy $\theta_f : \mathrm{Inv}(N, f) \to \Sigma_A$ onto the subshift of finite type on six symbols with the transition matrix*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

such that for every periodic $a \in \Sigma_A$ there exists a periodic point of f in $\theta_f^{-1}(a)$.

In particular, f has positive topological entropy on $\mathrm{Inv}(N, f)$.

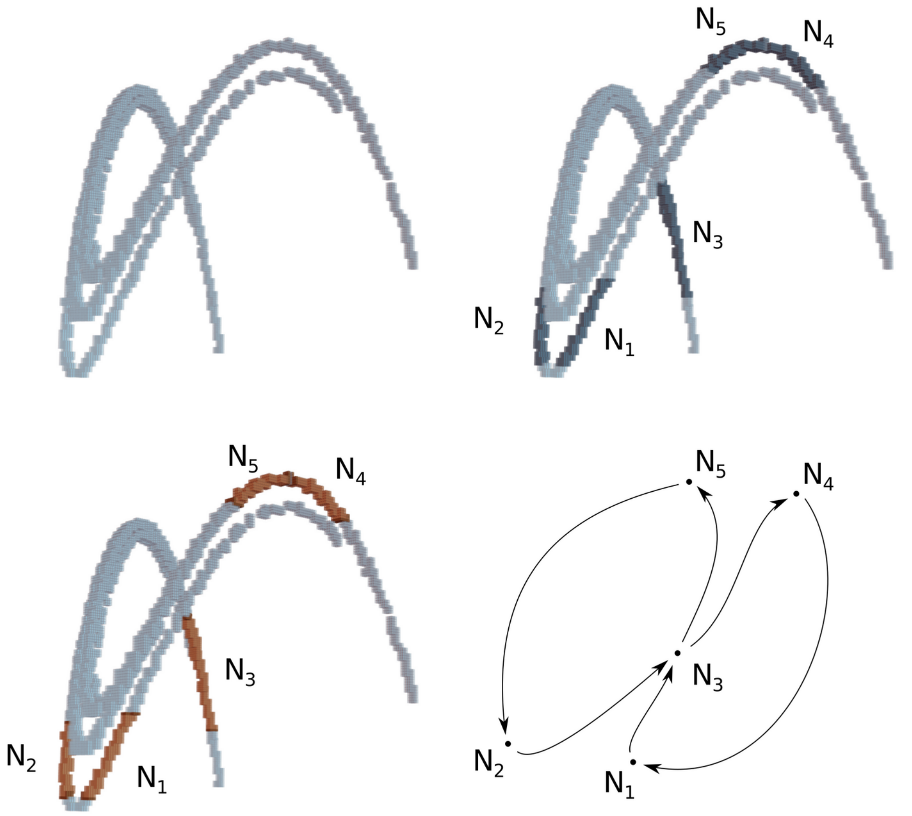


Figure 1.3. *Domain of sunflower enclosure for $g_{\bar{x}}$ consisting of 1029 3-dimensional cubes, an isolating neighborhood (in dark cyan), its weak index pair (in orange), and the graph of transitions between components of an isolating neighborhood.*

Note that in the above theorem, as well as in the oncoming theorems, we use a Hénon map with parameter values randomly selected from the set of values for which the system is chaotic.

Theorem 1.3. *Consider the time series* $\bar{x} = (x_i)_{i=100}^{14000}$ *generated by iterating the delayed Hénon map*

$$H : \mathbb{R}^3 \ni (x, y, z) \mapsto (1 - ax^2 + bz, x, y) \in \mathbb{R}^3$$

with the parameter values $a = 1.65$, $b = 0.1$, *and initial point* $(x_0, y_0, z_0) = (0, 0, 0)$. *Set*

$$A_{\bar{x}} := \{ (x_i, x_{i+1}, x_{i+2}) \mid i = 100, \ldots, 13{,}998 \}$$

and let $g_{\bar{x}} : A_{\bar{x}} \to \mathbb{R}^3$ *be given by* $g_{\bar{x}}(x_i, x_{i+1}, x_{i+2}) = (x_{i+1}, x_{i+2}, x_{i+3})$.

Choose a binning of $\mathbb{R}^3$ *based on* $\delta := 0.035256$ *and let* $F := F^{s}_{g_{\bar{x}},\delta} : K_\delta(A_{\bar{x}}) \subset \mathbb{R}^3$ *be the sunflower enclosure of* $g_{\bar{x}}$.

Let $\varepsilon = \delta/2$.

Then, $a_\varepsilon(F) \neq \emptyset$. *Furthermore, there exists a compact set* $N \subset \mathbb{R}^3$ *(see Figure 1.3) such that for any* $f \in a_\varepsilon(F)$

(i) *N is an isolating neighborhood of f; and*
(ii) *there exists a semiconjugacy $\theta_f : \mathrm{Inv}(N, f) \to \Sigma_A$ onto the subshift of finite type on five symbols with the transition matrix*

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

such that for every periodic $a \in \Sigma_A$ there exists a periodic point of f in $\theta_f^{-1}(a)$.
In particular, f has positive topological entropy on $\mathrm{Inv}(N, f)$.

The next theorem shows that our approach can also be successfully applied in the case of sparse data. Naturally, the dynamics that we capture is simpler than chaotic dynamics, but we employ significantly less data.

Theorem 1.4. *Consider the time series $\bar{x} = (x_i)_{i=0}^{286}$ generated by iterating the Hénon map*

$$H : \mathbb{R}^2 \ni (x, y) \mapsto (1 - ax^2 + by, x) \in \mathbb{R}^2$$

with the parameter values $a = 1.65$, $b = 0.1$, and initial condition

$$(x_0, y_0) = (0.891532, -0.346078).$$

Set

$$A_{\bar{x}} := \{ (x_i, x_{i+1}) \mid i = 0, \ldots, 285 \}$$

and let $g_{\bar{x}} : A_{\bar{x}} \to \mathbb{R}^2$ be given by $g_{\bar{x}}(x_i, x_{i+1}) = (x_{i+1}, x_{i+2})$.

Choose a binning of $\mathbb{R}^2$ based on $\delta := 0.036$ and let $F := F^s_{g_{\bar{x}},\delta} : K_\delta(A_{\bar{x}}) \rightrightarrows \mathbb{R}^2$ be the sunflower enclosure of $g_{\bar{x}}$. Let $\varepsilon = \delta/2$.

Then, $a_\varepsilon(F) \neq \emptyset$. Furthermore, there exists a compact set $N \subset \mathbb{R}^3$ (see Figure 1.4) such that for any $f \in a_\varepsilon(F)$ the set N is an isolating neighborhood of f and f has a 2-periodic point in $\mathrm{Inv}(N, f)$.

We run experiments relating to a periodic orbit for larger numbers of data points: 300, 500, and 1000. In each case we obtained the same conclusion as in Theorem 1.4. For attempts with less than 286 points the construction of the multivalued representation failed: our algorithm reported the existence of nonacyclic values of the enclosure.

A similar comment applies for Theorems 1.2 and 1.3. We verified that one can derive the same conclusions using time series consisting of 25000 or 30000 elements. We were unable to go below 20689 points in Theorem 1.2. One can verify the conclusion of Theorem 1.3 using time series with less than 14000 elements, however, we did not try to find the minimal length of the time series.

Since, till now, we use data obtained by sampling the system whose generator is known, we can compare the obtained results with the dynamics generated by the generator map itself. To this end, we use interval methods [28] to obtain a rigorous combinatorial enclosure of the Hénon map. Then we apply methods presented in [24]. In particular, we can prove the following theorem.

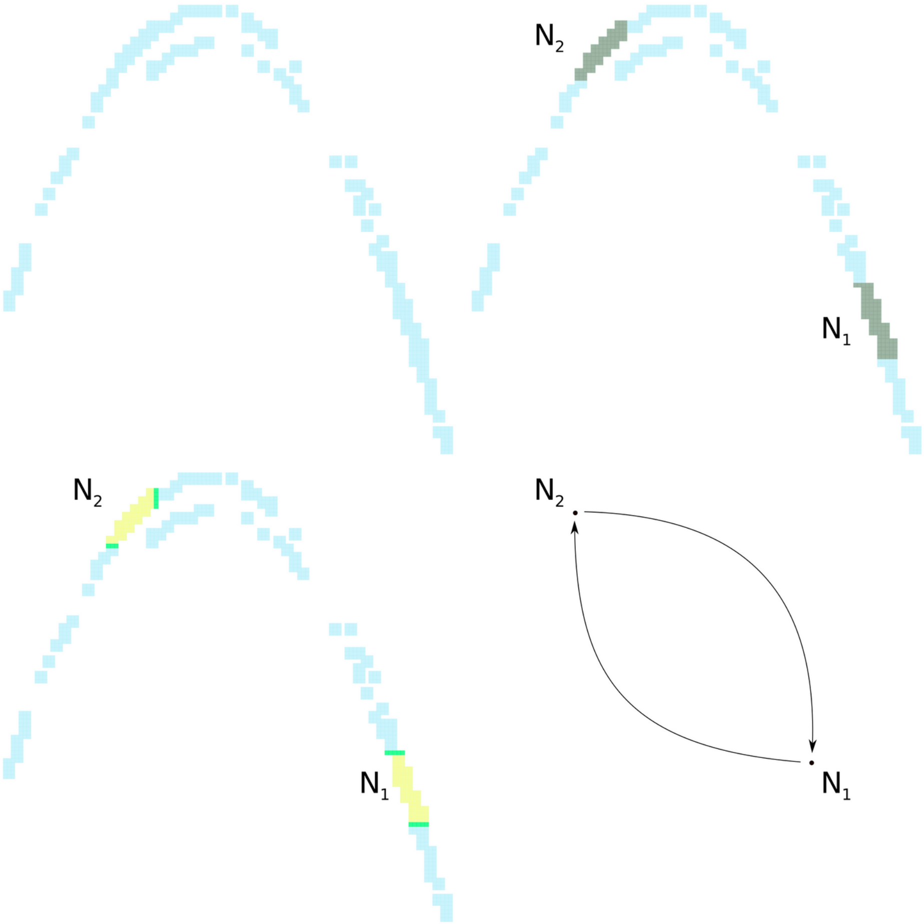


Figure 1.4. *Domain of sunflower enclosure for $g_{\bar{x}}$ consisting of 106 2-dimensional cubes, an isolating neighborhood (in dark sea green), its weak index pair (P_1 in yellow, P_2 in green), and the graph of transitions between components of an isolating neighborhood. Lower dimensional cubes are enlarged to 2-dimensional cubes.*

Theorem 1.5. *Consider the Hénon map*

$$H : \mathbb{R}^2 \ni (x, y) \mapsto (1 - ax^2 + by, x) \in \mathbb{R}^2$$

with the parameter values $a = 1.65$, $b = 0.1$. Let N be given by Theorem 1.4. Then, $\{(x, y) \in \mathbb{R}^2 \mid (y, x) \in N\}$ is an isolating neighborhood for H, isolating the 2-periodic orbit of H.

We note that in the presented examples the dynamics of the shift maps is expanding. In particular, in Theorem 1.2 the Lipschitz constant of the shift map $g_{\bar{x}}$ is estimated to be 3.70067, in Theorem 1.3 it is estimated to be 3.6327, and in Theorem 1.4 it is estimated to be 3.43745.

Theorems 1.2–1.4 are only meant to illustrate the proposed method. In this paper we focus on the theoretical results needed for the method. Several questions have to be addressed to make the method work in concrete problems. In particular, a question arises of how sensitive these results are to the choice of δ, the length of the time series, or the choice of initial condition. The fundamental feature of the Conley index is that it does not change under a small perturbation of the generator of the dynamical systems. Thus, the question reduces to the understanding of the stability of the multivalued map representation of the data. It is natural to expect that by increasing the length of the time series or changing the initial condition the semiconjugacy should be preserved as long as the same isolating neighborhood is used. Experiments we run confirm this expectation. More delicate is the question how the choice of δ affects the results. On one hand, if δ is very small, then the domain of the multivalued representation becomes a collection of isolated cubes. Therefore, it cannot properly approximate the phase space which is a continuum. On the other hand, if δ is too large, the multivalued representation gives a very coarse description of dynamics. Therefore, one cannot expect that it will give an interesting description of dynamics. Thus, the optimum is somewhere in the middle. Experiments we run show that small changes to δ preserve the results and moderate changes lead to a different matrix A but still let us claim the existence of an invariant set with positive entropy. An interesting problem is to get the understanding of changes in the results under varying δ in the spirit of persistent homology. All these practical questions are left for future investigations.

We now provide an outline for the paper. Section 2 provides basic definitions related to the Conley index. Section 3 presents results about isolating neighborhoods in the context of upper semicontinuous multivalued maps. Section 4 makes use of the results of section 3 to provide conditions under which continuous functions in a neighborhood of the graph of an upper semicontinuous multivalued map F with convex compact images inherit isolating neighborhoods and their associated Conley index from F. Results of this form are essential. The isolating neighborhood and Conley index computations in Theorems 1.2, 1.3, and 1.4 are done using the sunflower enclosure F, but the results of interest concern the dynamics generated by continuous functions in $a_\varepsilon(F)$.

The conclusions of Theorems 1.2 and 1.3 involve the existence of a semiconjugacy. As indicated above this is done via the Conley index. Because we work with upper semicontinuous multivalued maps that need not admit a continuous selector, we need to work with *weak index pairs*. The classical result of Szymczak [34, 35, 36] that proves the existence of a semiconjugacy onto symbolic dynamics is based on a stronger definition of an index pair and therefore cannot be applied directly. Section 9 presents theorems that are an extension of Szymczak's results. Sections 6–8 provide the necessary background to prove the results of section 9.

The fact that Theorems 1.2 and 1.3 contain explicit bounds on the class of maps, e.g., $a_\varepsilon(F)$ with $\varepsilon = \delta/2$ is important for the development of models. Section 5 provides explicit information about the preservation of topological and dynamical properties for continuous functions near F.

Finally, the proofs of Theorems 1.2, 1.3, and 1.4 are presented in section 10.

2. Preliminaries.

Throughout this paper by an interval in the set of integers Z we mean the intersection of a closed interval in R with Z. For $n \geq 1$ let $I_n := \{1, 2, \ldots, n\}$ and for $p \geq 2$ let $\mathbb{Z}_p := \{0, 1, \ldots, p-1\}$ denote the additive topological group with addition modulo p and discrete topology.

Given a topological space X and a subset $A \subset X$, by $\mathrm{int}_X A$, $\mathrm{cl}_X A$ we denote the *interior* of A in X and the *closure* of A in X, respectively. We omit the symbol of space if the space is clear from the context.

Let X, Y be topological spaces. By $F : X \multimap Y$ we denote a multivalued map, that is, a map $F : X \ni x \mapsto F(x) \in \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ is the power set of Y. A multivalued map F is *upper semicontinuous* if for any closed $B \subset Y$ its large counterimage under F, that is, the set $F^{-1}(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}$, is closed.

Throughout the paper we identify F with its *graph*, the set $\{(x, y) \in X \times Y \mid y \in F(x)\}$. In the following, we are interested in multivalued self-maps, that is, multivalued maps of the form $F : X \multimap X$.

Let I be an interval in $\mathbb{Z}$ with $0 \in I$. A single valued mapping $\sigma : I \to X$ is a *solution for F through $x \in X$* if $\sigma(n + 1) \in F(\sigma(n))$ for all $n, n + 1 \in I$ and $\sigma(0) = x$ (cf. [19, Definition 2.3]). Given a subset $N \subset X$, the set

$$\mathrm{Inv}(N, F) := \{x \in N \mid \exists \sigma : \mathbb{Z} \to N \text{ a solution for } F \text{ through } x\}$$

is called the *invariant part* of N. A compact subset $N \subset X$ is an *isolating neighborhood* for F if $\mathrm{Inv}(N, F) \subset \mathrm{int}\, N$. A compact subset $N \subset X$ is called an *isolating block* with respect to F if

$$N \cap F(N) \cap F^{-1}(N) \subset \mathrm{int}\, N.$$

Note that any isolating block is an isolating neighborhood. A compact set $S \subset X$ is said to be *invariant* with respect to F if $S = \mathrm{Inv}(S, F)$. It is called an *isolated invariant set* if it admits an isolating neighborhood N for F such that $S = \mathrm{Inv}(N, F)$ (cf. [2, Definition 4.1, Definition 4.3]).

By F-*boundary* of a given set $A \subset X$ we mean $\mathrm{bd}_F A := \mathrm{cl}\, A \cap \mathrm{cl}(F(A) \setminus A)$.

Definition 2.1 (cf. [2, Definition 4.7]). *Let $N \subset X$ be an isolating neighborhood for F. A pair $P = (P_1, P_2)$ of compact sets $P_2 \subset P_1 \subset N$ is called a* weak index pair *in N if*

(a) $F(P_i) \cap N \subset P_i$ *for* $i \in \{1, 2\}$;
(b) $\mathrm{bd}_F P_1 \subset P_2$;
(c) $\mathrm{Inv}(N, F) \subset \mathrm{int}(P_1 \setminus P_2)$;
(d) $P_1 \setminus P_2 \subset \mathrm{int}\, N$.

A set $B \subset X$ is *acyclic* if it has the (co)homology of a point. The multivalued map $F : X \multimap X$ is acyclic if it has acyclic values, that is, if for each $x \in X$ the set $F(x)$ is acyclic.

Given a weak index pair P in an isolating neighborhood $N \subset X$ for F we set

$$T_N(P) := (T_{N,1}(P), T_{N,2}(P)) := (P_1 \cup (X \setminus \mathrm{int}\, N), P_2 \cup (X \setminus \mathrm{int}\, N)).$$

Recall (cf., e.g., [2, 24]) that F_P, the restriction of F to the domain P, is a multivalued map of pairs, $F_P : P \multimap T_N(P)$; the inclusion $i_P : P \to T_N(P)$ induces an isomorphism in the Alexander–Spanier cohomology; and the *index map* I_{F_P} is defined as an endomorphism of $H^*(P)$ given by

$$I_{F_P} = F_P^* \circ (i_P^*)^{-1}.$$

The pair $(H^*(P), I_{F_P})$ is a graded module equipped with an endomorphism. Applying the Leray functor L (cf. [25, 2]) to $(H^*(P), I_{F_P})$ we obtain a graded module with its endomorphism which we call the *Leray reduction of the Alexander–Spanier cohomology of a weak index pair* P.

Definition 2.2 (cf. [2, Definition 6.3]). *The graded module $L(H^*(P), I_{F_P})$, that is, the Leray reduction of the Alexander–Spanier cohomology of a weak index pair P is called the* cohomological Conley index of $\mathrm{Inv}(N, F)$ *and denoted by* $C(\mathrm{Inv}(N, F), F)$.

3. Dynamics of upper semicontinuous maps.

Let (X, d) be a metric space. By $B_r(x)$ we denote the open ball with the center in $x \in X$ and radius $r > 0$. Closed balls will be denoted by $\overline{B}_r(x)$. For a given $A \subset X$, $B_r(A)$ will stand for an *open r-hull of* A, that is,

$$B_r(A) := \bigcup \{B_r(a) \mid a \in A\}.$$

Let $F : X \multimap X$ be an upper semicontinuous map. One can easily verify that (multivalued) selections of F share with F its isolating neighborhood and a weak index pair. We express this observation here for further reference.

Proposition 3.1. *Assume N is an isolating neighborhood for an upper semicontinuous $F : X \multimap X$, P is a weak index pair for F in N, and $G : X \multimap X$ is an upper semicontinuous map such that $G \subset F$. Then N is an isolating neighborhood for G, and P is a weak index pair for G in N.*

The aim of this section is to show that, to a certain extent, the reverse implications hold true. To be precise, we have the following theorem.

Theorem 3.2. *Let N be an isolating neighborhood with respect to an upper semicontinuous map $F : X \multimap X$. There exists an $\varepsilon > 0$ such that N is an isolating neighborhood with respect to an arbitrary upper semicontinuous map $G : X \multimap X$ with $G \subset B(F, \varepsilon)$.*

We postpone its proof to the end of this section.

Lemma 3.3. *Let $A \subset X$ be a compact set and let $\{x_n\} \subset X$ be a sequence convergent to $x \in X$. If $x_n \in B(A, \frac{1}{n})$ for $n \in \mathbb{N}$ then $x \in A$.*

Proof. Suppose the contrary and consider an $r > 0$ such that $B(x, r) \cap A = \emptyset$. Observe that, for large enough $n \in \mathbb{N}$, we have $d(x_n, x) \leq \frac{r}{2}$. Moreover, there exists a sequence $\{u_n\} \subset A$ with $d(u_n, x_n) \leq \frac{1}{n}$ for $n \in \mathbb{N}$. However, $d(x_n, u_n) \geq d(u_n, x) - d(x_n, x) \geq r - \frac{r}{2} = \frac{r}{2}$, a contradiction. ∎

Lemma 3.4. *Let $F : X \multimap X$ be upper semicontinuous and let $N \subset X$ be compact. A solution $\tau : \mathbb{Z} \to N$ for F through $x \in N$ exists provided for any $n \in \mathbb{N}$ there exists a solution $\sigma : [-n, n] \to N$ through x.*

Proof. Let $\sigma^n : [-n, n] \to N$ be a solution with respect to F through x. By induction we construct a sequence of solutions $\tau^n : [-n, n] \to N$ for F through x such that

(p1) there exists a strictly increasing sequence $\{m_p\} \subset \mathbb{N}$ such that $\tau^n(k) = \lim_{p\to\infty} \sigma^{m_p}(k)$ for any $k \in [-n, n]$;

(p2) $\tau^{n-1} \subset \tau^n$ for $n \geq 1$.

Define $\tau^0 : [0] \to N$ by putting $\tau^0(0) := x$. Clearly (p1) and (p2) hold. Suppose τ^n has been constructed so that (p1) and (p2) hold. Denote $\overline{\sigma}^p := \sigma^{m_p}$ and take into account a subsequence $\bar{p}$ such that the sequences $\bar{\sigma}^{\bar{p}}(n+1)$ and $\bar{\sigma}^{\bar{p}}(-n-1)$ converge to $v, w \in N$, respectively. We define $\tau^{n+1} : [-n-1, n+1] \to N$ by

$$\tau^{n+1}(k) := \begin{cases} \tau^n(k), & |k| \le n, \\ v, & k = n+1, \\ w, & k = -n-1. \end{cases}$$

It is straightforward to see that conditions (p1) and (p2) hold, and $\tau^{n+1}(0) = x$. It remains to be verified that τ^{n+1} is a solution for F. Since $\bar{\sigma}^{\bar{p}}$ is a solution for F, we have

$$\bar{\sigma}^{\bar{p}}(k+1) \in F(\bar{\sigma}^{\bar{p}}(k)), \; k \in \mathbb{Z}. \tag{2}$$

For any $k \in [-n-1, n+1]$ the sequence $\bar{\sigma}^{\bar{p}}(k)$ converges to $\tau^{n+1}(k)$. Because the graph of F is closed (cf. [14, Proposition 14.4]), passing to the limit in (2) we have $\tau^{n+1}(k+1) \in F(\tau^{n+1}(k))$. ∎

Proof of Theorem 3.2. For contradiction suppose that for any $m \in \mathbb{N}$ there exists an upper semicontinuous $G_m : X \multimap X$ with $G_m \subset B(F, \frac{1}{m})$ and such that $\operatorname{Inv}(N, G_m) \cap \operatorname{bd} N \neq \varnothing$. Let $x_m \in \operatorname{Inv}(N, G_m) \cap \operatorname{bd} N$. Passing to a subsequence, if necessary, we may assume that x_m converges to an $x \in \operatorname{bd} N$. Let $\sigma_m : \mathbb{Z} \to N$ be a solution for G_m through x_m. Fix an integer $n \in \mathbb{N}$, choose a subsequence m_p such that for any $k \in [-n, n]$ the sequence $\sigma_{m_p}(k)$ is convergent, and define $\tau^n : [-n, n] \to N$ by putting $\tau^n(k) := \lim_{p\to\infty} \sigma_{m_p}(k)$ for $k \in [-n, n]$. We have $(\sigma_{m_p}(k), \sigma_{m_p}(k+1)) \in G_{m_p} \subset B(F, \frac{1}{m_p})$. Using Lemma 3.3 we infer that $(\tau^n(k), \tau^n(k+1)) \in F$, which means that $\tau^n : [-n, n] \to N$ is a solution for F through x. This, along with Lemma 3.4, yields the existence of a solution $\tau : \mathbb{Z} \to N$ for F through x. However, $x \in \operatorname{bd} N$, a contradiction. ∎

4. ε-Approximations.

In the following we consider the Cartesian product of normed spaces as the normed space with the max-norm.

Following [15] (cf., e.g., [14]) we say that a continuous single valued map $f : X \to X$ is a continuous ε-approximation (on the graph) of $F : X \multimap X$ if $f \subset B_\varepsilon(F)$. We denote the set of continuous ε-approximations of F by $a_\varepsilon(F)$.

Theorem 4.1. *Let Y be a normed space and let $X \subset Y$ be compact. Assume that $F : X \multimap X$ is an upper semicontinuous map with convex and compact values, and N is an isolating neighborhood with respect to F. Then*

(i) *there exists an $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon \le \varepsilon_0$, there is a continuous ε-approximation $f : X \to X$ of F such that N is an isolating neighborhood with respect to f, and $C(\operatorname{Inv}(N, F), F) = C(\operatorname{Inv}(N, f), f)$;*

(ii) *if X is an absolute neighborhood retract (ANR) then there exists a $\delta > 0$ such that for any continuous δ-approximation $g : X \to X$ of F we have $C(\operatorname{Inv}(N, F), F) = C(\operatorname{Inv}(N, g), g)$.*

Proof. Take an $\varepsilon_0 > 0$ as in Theorem 3.2 and $0 < \varepsilon \le \varepsilon_0$. By [6, Theorem 1] there exists a continuous ε-approximation $f : X \to X$ of F. We shall prove that f satisfies the assertions (i) and (ii).

To this end, for $\lambda \in [0, 1]$, we define $F_\lambda : X \multimap X$ by

$$F_\lambda(x) := \lambda f(x) + (1-\lambda)F(x), \; x \in X.$$

It follows from the upper semicontinuity of F and the continuity of f that F_λ is upper semicontinuous and it is straightforward to observe that F_λ has convex and compact values.

According to the construction of the ε-approximation f of F in [6], for arbitrarily fixed $x \in X$ there exists an $x^0 \in B_\varepsilon(x)$ such that $f(x) \in B_\varepsilon(F(x^0))$ and $F(x) \subset B_\varepsilon(F(x^0))$. Therefore, for any $\lambda \in [0, 1]$, we have $F_\lambda(x) \subset B_\varepsilon(F(x^0))$, as $B_\varepsilon(F(x^0))$ is convex. Consequently, $F_\lambda \subset B_\varepsilon(F)$. Theorem 3.2 shows that N is an isolating neighborhood with respect to F_λ for every $\lambda \in [0, 1]$. Therefore, by the continuation property of the Conley index (cf. [1, Theorem 6.1]), we have $C(\mathrm{Inv}(N, F), F) = C(\mathrm{Inv}(N, f), f)$.

Let an $\varepsilon > 0$ be as above. By [14, Theorem 23.9] there is a $\delta \in (0, \varepsilon]$ such that for any $f, g : X \to X$, the δ-approximations of F, there exists a homotopy $h : X \times [0, 1] \to X$ joining f and g, such that $h(\cdot, t)$ is an ε-approximation of F, for all $t \in [0, 1]$. Fix such a $\delta > 0$ and consider $f : X \to X$, a δ-approximation of F defined as in [6]. Let $g : X \to X$ be an arbitrary δ-approximation of F. Since $\delta \le \varepsilon$, by [15, Theorem 5.13] (cf., e.g., [14, Theorem 23.9]) and Theorem 3.2, $\mathrm{Inv}(N, f)$ and $\mathrm{Inv}(N, g)$ are related by continuation; hence $C(\mathrm{Inv}(N, g), g) = C(\mathrm{Inv}(N, f), f)$. This, along with property (i), completes the proof. ■

5. ε-Approximations of cubical maps. In this section we assume that $X \subset \mathbb{R}^d$ is a closed subset and $F : X \multimap X$ is a multivalued cubical map (cf., e.g., [18]), a scale $\delta > 0$ responsible for δ-cubes is fixed, and $\%$ stands for the max metric in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ by σ_x we denote the unique elementary cube such that $x \in \mathring{\sigma}_x$. For $\varepsilon > 0$ define maps $F_\varepsilon, F^\varepsilon : X \multimap X$ by

$$F_\varepsilon(x) := F(\bar{B}_\varepsilon(x)) \tag{3}$$

and

$$F^\varepsilon(x) := \bar{B}_\varepsilon(F(x)). \tag{4}$$

We refer to maps F_ε and F^ε as a *horizontal* and a *vertical enclosure* of F, respectively.

We begin with some auxiliary lemmas.

Lemma 5.1. *Assume $A_1, A_2 \subset X$ are cubical, $\varepsilon \in (0, \frac{1}{2}\delta)$, and $y \in \bar{B}_\varepsilon(A_1) \cap \bar{B}_\varepsilon(A_2)$. Then, there exists a $y^0 \in A_1 \cap A_2$ such that $\%(y, y^0) \le 2\varepsilon$.*

Proof. For $i = 1, 2$ let $y_i \in A_i$ be such that $\%(y_i, y) < \varepsilon$. Then $\sigma_{y_1} \cap \sigma_{y_2} \neq \emptyset$ and $\sigma_{y_i} \subset A_i$. Let $y^0 \in \sigma_{y_1} \cap \sigma_{y_2}$. Then $y^0 \in A_1 \cap A_2$ and $\%(y, y^0) \le \%(y, y_1) + \%(y_1, y^0) \le 2\varepsilon$. ■

Lemma 5.2. *Assume $P \subset M \subset \mathbb{R}^d$ are cubical and $0 < \varepsilon < \frac{1}{2}\delta$. Then the inclusion $\mu : P \cup M \to \bar{B}_\varepsilon(P) \cup M$ induces an isomorphism in cohomology.*

Proof. Consider the multivalued map $G : \bar{B}_\varepsilon(P) \cup M \to M$ given by

$$G(x) := \{y \in M \mid \%(x, y) = \%(x, M)\}.$$

This map has compact values and is upper semicontinuous (see [27, Lemma 1]). Since $G(x) = \{x\}$ for $x \in M$, we see that $G \circ \mu = \mathrm{id}_{P \cup M}$. We will show that $\mu \circ G$ is homotopic to $\mathrm{id}_{\bar{B}_\varepsilon(P) \cup M}$. One easily verifies that $G(x) = \bar{B}_{\%(x,M)}(x) \cap M$. Let

$$\mathcal{Q} := \{Q \in \mathcal{K} \mid Q \subset M,\ Q \cap \bar{B}_{\%(x,M)}(x) \neq \emptyset\}$$

and $\mathcal{Q}' := \{Q \cap \bar{B}_\varepsilon(x) \mid Q \in \mathcal{Q}\}$. Then $G(x) = \bigcup_{Q' \in \mathcal{Q}'} Q'$. Each $Q' \in \mathcal{Q}'$ is a rectangle as an intersection of rectangles. Hence it is convex. We claim that $\bigcap_{Q' \in \mathcal{Q}'} Q' \neq \emptyset$. For this end it suffices to show that $Q'_1 \cap Q'_2 \neq \emptyset$ for any $Q'_1, Q'_2 \in \mathcal{Q}'$. Since $Q'_i \neq \emptyset, i \in \{1, 2\}$, take $x_i \in Q'_i$. Then $x_1, x_2 \in \bar{B}_\varepsilon(x)$, which means that $\varrho(x_1, x_2) \leq 2\varepsilon$. Since $\varepsilon < \frac{1}{2}\delta$, this implies that $Q'_1 \cap Q'_2 \neq \emptyset$. Thus, $G(x)$ is star-shaped (cf. [18, Definition 2.82]), hence, acyclic.

For $\lambda \in [0, 1]$ let

$$G_\lambda(x) := \{(1 - \lambda)x + \lambda y \mid y \in G(x)\}$$

and $D(x) := \bigcup_{\lambda \in [0,1]} G_\lambda(x)$. Note that if $x \in M$ then $G_\lambda(x) = D(x) = \{x\}$. Also if $x \in \bar{B}_\varepsilon(P) \cup M$ then $D(x) \subset \bar{B}_\varepsilon(P) \cup M$. Therefore,

$$[0, 1] \times (\bar{B}_\varepsilon(P) \cup M) \ni (\lambda, x) \longmapsto G_\lambda(x) \subset \bar{B}_\varepsilon(P) \cup M$$

is the requested homotopy between $\mu \circ G$ and $\mathrm{id}_{\bar{B}_\varepsilon(P) \cup M}$. ■

As a consequence of the previous lemma we have the following lemma.

Lemma 5.3. *Assume $A \subset \mathbb{R}^d$ is a cubical set and $0 < \varepsilon < \frac{1}{2}\delta$. Then A and $\bar{B}_\varepsilon(A)$ are homotopy equivalent.*

Now we enumerate a few properties of the enclosures.

Lemma 5.4. *The map F_ε has the following properties:*

(i) *If $A \subset X$ then $F_\varepsilon^{-1}(A) = \bar{B}_\varepsilon(F^{-1}(A))$.*

(ii) *If F is upper semicontinuous then so is F_ε.*

(iii) *If $\varepsilon < \frac{1}{2}\delta$ and F is upper semicontinuous then for any $x \in X$ there is $y \in X$ with $F(y) = F_\varepsilon(x)$.*

(iv) *If $\varepsilon < \frac{1}{2}\delta$ and F is upper semicontinuous and convex valued then so is F_ε.*

(v) *If $\varepsilon < \frac{1}{2}\delta$ and F is upper semicontinuous and has contractible values then so does F_ε.*

(vi) *If $\varepsilon < \frac{1}{2}\delta$ and F is an upper semicontinuous map with convex values then F_ε admits a continuous selection.*

(vii) *If $A \subset X$ is a cubical set and F is upper semicontinuous then $F_\varepsilon(A) = F(A)$ whenever $\varepsilon < \frac{1}{2}\delta$.*

Proof. In order to prove inclusion $\bar{B}_\varepsilon(F^{-1}(A)) \subset F_\varepsilon^{-1}(A)$ in (i) take an $x \in \bar{B}_\varepsilon(F^{-1}(A))$ and an $x' \in F^{-1}(A)$ such that $x \in \bar{B}_\varepsilon(x')$. Then $F(x') \cap A \neq \emptyset$. Take a $y \in F(x') \cap A$. Then $y \in F_\varepsilon(x)$ and $F_\varepsilon(x) \cap A \neq \emptyset$ which proves that $x \in F_\varepsilon^{-1}(A)$.

In the reverse direction, take an $x \in F_\varepsilon^{-1}(A)$, a $y \in F_\varepsilon(x) \cap A$, and an $x' \in \bar{B}_\varepsilon(x)$ such that $y \in F(x')$. It means that $F(x') \cap A \neq \emptyset$ and $x' \in F^{-1}(A)$. Therefore $x \in \bar{B}_\varepsilon(F^{-1}(A))$.

By (i), the large counterimage under F_ε of any closed set in its range is closed. Hence, F_ε is upper semicontinuous, and we have (ii).

In order to show (iii), fix $x \in X$ and consider the set

$$\mathcal{A}(x) := \left\{ \mathring{Q} \mid \mathring{Q} \text{ is a cell, } \mathring{Q} \cap \bar{B}_\varepsilon(x) \neq \emptyset \right\}.$$

Note that $\mathcal{A}(x) \neq \emptyset$. Since $\varepsilon < \frac{1}{2}\delta$, for any $\mathring{Q}, \mathring{Q}' \in \mathcal{A}(x)$ we have $Q \cap Q' \neq \emptyset$. Then $P := \bigcap_{\mathring{Q} \in \mathcal{A}(x)} Q \neq \emptyset$ and P is an elementary cube. Moreover, P is a face of every cube Q

with $\mathring{Q} \in A(x)$. Then, for any $y \in \mathring{P}$ we have $F(y) = F(\bar{B}_\varepsilon(x)) = F_\varepsilon(x)$, as F is upper semicontinuous.

Properties (iv) and (v) follow from (iii).

We shall prove (vi). To this end consider $\tilde{F}_\varepsilon : X \rightrightarrows X$ given by

$$\tilde{F}_\varepsilon(x) := F(B_\varepsilon(x)),\ x \in X.$$

It is easy to see, by the same reasoning as for F_ε, that $\tilde{F}_\varepsilon$ has nonempty convex and compact values. Moreover, the large counterimage under $\tilde{F}_\varepsilon$ of any single point in its range is open, hence, $\tilde{F}_\varepsilon$ is lower semicontinuous. Consequently, by Michael's selection theorem (cf., e.g., [21]), there exists a continuous map $f : X \to X$, a selection of $\tilde{F}_\varepsilon$. Clearly $\tilde{F}_\varepsilon \subset F_\varepsilon$, hence, f is a continuous selection of F_ε, as desired.

In order to prove (vii) take a $y \in F_\varepsilon(A)$. Then there exist an $x \in A$ such that $y \in F_\varepsilon(x)$ and an $x' \in \bar{B}(x, \varepsilon)$ such that $y \in F(x')$. Then $\sigma_x \cap \sigma_{x'} \neq \emptyset$, because $\rho(x, x') \le 2\varepsilon < \delta$. It follows that we can take an $x'' \in \sigma_x \cap \sigma_{x'}$. By the upper semicontinuity of F we set $F(x') \subset F(x'')$. Hence, $y \in F(x'') \subset F(\sigma_x) \subset F(A)$. The inclusion in the reverse direction is obvious. ■

Lemma 5.5. *The map F^ε has the following properties:*

(i) *If $A \subset X$ then $F^\varepsilon(A) = \bar{B}_\varepsilon(F(A))$.*

(ii) *If F is upper semicontinuous, then so is F^ε.*

(iii) *If F is convex valued then so is F^ε.*

(iv) *If $\varepsilon < \frac{1}{2}\delta$ and a cubical map F has contractible values then so does F^ε.*

(v) *If $A \subset X$ is a cubical set and F is cubical then $(F^\varepsilon)^{-1}(A) = F^{-1}(A)$ for any $0 \le \varepsilon < \delta$.*

Proof. To prove (i) observe that

$$\begin{aligned} F^\varepsilon(A) &= \bigcup_{x\in A} F^\varepsilon(x) = \bigcup_{x\in A} \bar{B}_\varepsilon(F(x)) \\ &= \bigcup_{x\in A} \bigcup_{y\in F(x)} \bar{B}_\varepsilon(y) = \bigcup_{y\in F(A)} \bar{B}_\varepsilon(y) = \bar{B}_\varepsilon(F(A)). \end{aligned}$$

Properties (ii) and (iii) are obvious.

Property (iv) is a consequence of Lemma 5.3.

In order to show inclusion $F^{-1}(A) \subset (F^\varepsilon)^{-1}(A)$ in (v) take $x \in F^{-1}(A)$. It means that $F(x) \cap A \neq \emptyset$ and $F^\varepsilon(x) \cap A \neq \emptyset$. Hence $x \in (F^\varepsilon)^{-1}(A)$.

To prove the opposite inclusion take an $x \in (F^\varepsilon)^{-1}(A)$. Since $F^\varepsilon(x) \cap A \neq \emptyset$, there exist a $y \in F^\varepsilon(x) \cap A$ and a $y' \in F(x)$ such that $y \in \bar{B}_\varepsilon(y')$. We have $\sigma_y \cap \sigma_{y'} \neq \emptyset$, because $\rho(y, y') \le \varepsilon < \delta$. Take $y'' \in \sigma_y \cap \sigma_{y'}$. Then $y'' \in \sigma_{y'} = \operatorname{cl} \mathring{\sigma}_{y'} \subset F(x)$, because F is cubical. Notice that $y \in \mathring{\sigma}_y \cap A$. Thus, $y'' \in \sigma_y \subset A$, because A is a cubical set. It follows that $F(x) \cap A \neq \emptyset$ and $x \in F^{-1}(A)$ which completes the proof. ■

Horizontal enclosures preserve isolating neighborhoods and weak index pairsMore precisely, we have the following propositions.

Proposition 5.6. *Assume $F : X \rightrightarrows X$ is a cubical, upper semicontinuous multivalued map and N is a cubical isolating neighborhood for F. Then, for any $\varepsilon < \delta$, we have* $\operatorname{Inv}(N, F_\varepsilon) \subset \bar{B}_\varepsilon(\operatorname{Inv}(N, F))$. *As a consequence, N is an isolating neighborhood for F_ε.*

Proof. Take an arbitrary $x_0 \in \mathrm{Inv}(N, F_\varepsilon)$ and consider $x : \mathbb{Z} \to N$, a solution for F_ε in N through x_0. Let $n \in \mathbb{Z}$ be fixed. We have $x_{n+1} \in F_\varepsilon(x_n)$. There exists an $x_n^0 \in N$ such that $F_\varepsilon(x_n) = F(x_n^0)$ and $\varrho(x_n^0, x_n) \le \varepsilon < \delta$. Therefore we can take an $x_n^{00} \in \sigma_{x_n} \cap \sigma_{x_n^0}$ such that $\varrho(x_n^{00}, x_n) \le \varepsilon$ and $F(x_n^0) \subset F(x_n^{00})$, as F is upper semicontinuous. We have $x_{n+1} \in F(x_n^0)$ and $\sigma_{x_{n+1}} \subset F(x_n^0)$, because F is cubical. Moreover, $x_{n+1}^{00} \in F(x_n^0) \subset F(x_n^{00})$ and $\varrho(x_n^{00}, x_n) \le \varrho(x_n^0, x_n) \le \varepsilon$. Since $n \in \mathbb{Z}$ was arbitrarily fixed, we have constructed $x^{00} : \mathbb{Z} \to N$, a solution for F in N with $\varrho(x_n^{00}, x_n) \le \varepsilon$. In particular, $x_0 \in \bar{B}_\varepsilon(\mathrm{Inv}(N, F))$, showing that $\mathrm{Inv}(N, F_\varepsilon) \subset \bar{B}_\varepsilon(\mathrm{Inv}(N, F))$.

Since $\mathrm{Inv}(N, F) \subset \mathrm{int}\, N$ and $\varepsilon < \delta$, the latter inclusion yields $\mathrm{Inv}(N, F_\varepsilon) \subset \bar{B}_\varepsilon(\mathrm{Inv}(N, F)) \subset \mathrm{int}\, N$. This completes the proof. ■

Proposition 5.7. *Assume $F : X \rightrightarrows X$ is a cubical, upper semicontinuous multivalued map, N is a cubical isolating neighborhood for F, P is a cubical weak index pair in N, and $\varepsilon < \frac{1}{2}\delta$. Then P is a weak index pair for F_ε.*

Proof. Properties (a) and (b) of Definition 2.1 are straightforward consequences of Lemma 5.4(vii).

By Theorem 5.6 we have $\mathrm{Inv}(N, F_\varepsilon) \subset \bar{B}_\varepsilon(\mathrm{Inv}(N, F)) \subset \mathrm{int}(P_1 \setminus P_2)$, and property (c) follows.

Property (d) is obvious. ■

Note that, in general, F_ε is not a cubical map. However, it inherits from F the following property.

Lemma 5.8. *If $F : X \rightrightarrows X$ is a cubical map and $\varepsilon < \frac{1}{2}\delta$, then*

$$F_\varepsilon(y) \subset F_\varepsilon(x) \text{ whenever } \sigma_x \subset \sigma_y. \tag{5}$$

Proof. Since $\varepsilon < \frac{1}{2}\delta$ and $\sigma_x \subset \sigma_y$, for an arbitrary elementary cube σ, condition $\sigma \cap \bar{B}_\varepsilon(y) \neq \emptyset$ implies $\sigma \cap \bar{B}_\varepsilon(x) \neq \emptyset$. Therefore, taking into account that F is cubical, we have

$$\begin{aligned}
F_\varepsilon(y) &= F(\bar{B}_\varepsilon(y)) \\
&= \bigcup_{z \in \bar{B}_\varepsilon(y)} F(z) \\
&= \bigcup_{\sigma \cap \bar{B}_\varepsilon(y) \neq \emptyset} F(\mathring{\sigma}) \\
&\subset \bigcup_{\sigma \cap \bar{B}_\varepsilon(x) \neq \emptyset} F(\mathring{\sigma}) \\
&= \bigcup_{z \in \bar{B}_\varepsilon(x)} F(z) \\
&= F(\bar{B}_\varepsilon(x)) \\
&= F_\varepsilon(x).
\end{aligned}$$

This completes the proof. ■

Proposition 5.9. *Assume $F : X \rightrightarrows X$ is a cubical, upper semicontinuous multivalued map and N is a cubical isolating neighborhood for F. Then N is an isolating neighborhood for $(F_\varepsilon)^\varepsilon$ for any $\varepsilon < \frac{1}{2}\delta$.*

Proof. For contradiction, suppose that $x_0 \in \operatorname{bd} N$ and $x : \mathbb{Z} \to N$ is a solution for $(F_\varepsilon)^\varepsilon$ in N through x_0.

Let $n \in \mathbb{Z}$ be fixed. We have $x_{n+1} \in (F_\varepsilon)^\varepsilon(x_n)$. There exists an $x'_{n+1} \in F_\varepsilon(x_n)$ with $\rho(x'_n, x_n) \leq \varepsilon < \frac{1}{2}\delta$. Then $\sigma_{x_{n+1}} \cap \sigma_{x'_{n+1}} \neq \emptyset$ and we can take $x''_{n+1} \in \sigma_{x_{n+1}} \cap \sigma_{x'_{n+1}}$. Since N is cubical and $x_{n+1} \in N$, we infer that $\sigma_{x_{n+1}} \subset N$. Hence, $x''_{n+1} \in \sigma_{x_{n+1}} \cap \sigma_{x'_{n+1}} \subset N$. Similarly, $x''_{n+1} \in \sigma_{x_{n+1}} \cap \sigma_{x'_{n+1}} \subset F_\varepsilon(x_n)$, as $x'_{n+1} \in F_\varepsilon(x_n)$ and $F_\varepsilon(x_n)$ is a cubical set. By Lemma 5.8 we have $F_\varepsilon(x_n) \subset F_\varepsilon(x''_n)$. This, along with $x''_{n+1} \in F_\varepsilon(x_n)$ yields $x''_{n+1} \in F_\varepsilon(x''_n)$. Since $n \in \mathbb{Z}$ was arbitrarily fixed, we have defined $x'' : \mathbb{Z} \to N$, a solution with respect to F_ε in N.

Note that $x''_0 \in \operatorname{bd} N$, because $x_0 \in \operatorname{bd} N$, $x''_0 \in \sigma_{x_0}$, and $\operatorname{bd} N$ is a cubical set. This contradicts Proposition 5.6, and completes the proof. ∎

The following theorem is a counterpart of Theorem 4.1 for cubical maps.

Theorem 5.10. *Assume that $F : X \rightrightarrows X$ is an upper semicontinuous cubical map with contractible values and $\varepsilon < \frac{1}{2}\delta$. Then $a_\varepsilon(F) \neq \emptyset$. Moreover, if N is a cubical isolating neighborhood with respect to F then N is an isolating neighborhood with respect to arbitrary $f \in a_\varepsilon(F)$, and $C(\operatorname{Inv}(N, f), f) = C(\operatorname{Inv}(N, F), F)$.*

Proof. The existence of an ε-approximation $f : X \to X$ of F follows from [15, Theorem 5.12] (cf., e.g., [14, Theorem 23.8]).

Since F has contractible values then, by Lemma 5.4(v) and Lemma 5.5(iv), so does $(F_\varepsilon)^\varepsilon$. Moreover, by Proposition 5.9, N is an isolating neighborhood with respect to $(F_\varepsilon)^\varepsilon$. Therefore we have a well-defined Conley index $C(\operatorname{Inv}(N, (F_\varepsilon)^\varepsilon), (F_\varepsilon)^\varepsilon)$. Since, in addition $F \subset (F_\varepsilon)^\varepsilon$, we infer that $C(\operatorname{Inv}(N, F), F) = C(\operatorname{Inv}(N, (F_\varepsilon)^\varepsilon), (F_\varepsilon)^\varepsilon)$. Note that if $f : X \to X$ is an ε-approximation of F, then we have $f \subset (F_\varepsilon)^\varepsilon$, and the identity $C(\operatorname{Inv}(N, f), f) = C(\operatorname{Inv}(N, (F_\varepsilon)^\varepsilon), (F_\varepsilon)^\varepsilon)$ follows. This completes the proof. ∎

A statement analogous to Proposition 5.7 for map F_ε is not true, however, an approximate version holds.

Theorem 5.11. *Assume that $F : X \rightrightarrows X$ is an upper semicontinuous map with cubical values satisfying (5), P is a cubical weak index pair with respect to F in a cubical isolating neighborhood N, and $\varepsilon < \frac{1}{2}\delta$. Then $\bar{B}_\varepsilon(P)$ is a weak index pair for F^ε in $\bar{B}_\varepsilon(N)$.*

Proof. For the proof of property (a) in Definition 2.1 fix an $i \in \{1, 2\}$ and take an $x \in \bar{B}_\varepsilon(P_i)$ and a $y \in F^\varepsilon(x) \cap \bar{B}_\varepsilon(N)$. Then, there exists an $x' \in P_i$ such that $\rho(x, x') < \varepsilon$ and by Lemma 5.1 there exists a $y' \in F(x) \cap N$ such that $\rho(y, y') < 2\varepsilon < \delta$. Then $\sigma_x \cap \sigma_{x'} \neq \emptyset$ and we can take an $x'' \in \sigma_x \cap \sigma_{x'}$ such that $\rho(x, x'') < \varepsilon$. By (5) we have $F(x'') \supset F(x)$ and since P is cubical, we have $x'' \in \tau \subset P_i$. Similarly, $\rho(y, y') < 2\varepsilon < \delta$ implies that there exists a $y'' \in \sigma_y \cap \sigma_{y'}$ such that $\rho(y, y'') < \varepsilon$. Since $F(x)$ and N are cubical and $y' \in \mathring{\sigma}_{y'} \cap F(x) \cap N$, we get $\sigma_{y'} \subset F(x) \cap N$. Therefore $y'' \in F(x) \cap N \subset F(x'') \cap N$. By property (a) of P, we have $y'' \in P_i$. Hence, $y \in \bar{B}_\varepsilon(P_i)$.

In order to prove property (b) assume the contrary. Let $x \in \operatorname{bd}_{F^\varepsilon} \bar B_\varepsilon(P_1) \setminus \bar B_\varepsilon(P_2)$. It means that $x \in \bar B_\varepsilon(P_1)$, $x \in \operatorname{cl}(F^\varepsilon(\bar B_\varepsilon(P_1)) \setminus \bar B_\varepsilon(P_1))$, and $x \notin \bar B_\varepsilon(P_2)$. Take an $x' \in P_1$ such that $\varrho(x, x') < \varepsilon$. Then $x' \notin P_2$, that is $x' \in P_1 \setminus P_2$. Consider a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in F^\varepsilon(\bar B_\varepsilon(P_1)) \setminus \bar B_\varepsilon(P_1)$ and $x_n \to x$. It follows that for every $n \in \mathbb{N}$ we have $x_n \in F^\varepsilon(u_n)$ for some $u_n \in \bar B_\varepsilon(P_1)$. Take a $u_n' \in P_1$ such that $\varrho(u_n, u_n') < \varepsilon$ and a $z_n \in F(u_n)$ such that $\varrho(x_n, z_n) < \varepsilon$. We have $z_n \notin P_1$, because otherwise $x_n \in \bar B_\varepsilon(P_1)$. By $\varrho(u_n, u_n') < \varepsilon$, we can take $u_n'' \in \sigma_{u_n} \cap \sigma_{u_n'}$. Since P is cubical, we have $u_n'' \in P_1$. Since F is cubical, we have $F(u_n'') \supset F(u_n)$. Hence, $z_n \in F(P_1) \setminus P_1$. Without loss of generality we may assume that $z_n \to z \in \operatorname{cl}(F(P_1) \setminus P_1)$. Since

$$\varrho(z, x') \le \varrho(z, x) + \varrho(x, x') \le 2\varepsilon < \delta,$$

we can find $\bar z \in \sigma_z \cap \sigma_{x'} \cap \sigma_x$ with $\varrho(x, \bar z) \le \varepsilon$. We have $\bar z \in \operatorname{cl}(F(P_1) \setminus P_1) \cap P_1$, because $\mathring{\sigma}_z \subset \operatorname{cl}(F(P_1) \setminus P_1)$ and $\sigma_{x'} \subset P_1$. By property (b) of P, $\bar z \in P_2$. It follows that $x \in \bar B_\varepsilon(P_2)$, a contradiction.

We shall prove that

(6) $$\operatorname{Inv}(\bar B_\varepsilon(N), F^\varepsilon) \subset \bar B_\varepsilon(\operatorname{Inv}(N, F)).$$

Let $x\colon \mathbb{Z} \to \bar B_\varepsilon(N)$ be a solution for F^ε in $\bar B_\varepsilon(N)$. For each $x_i \in \bar B_\varepsilon(N)$, we can choose an $x_i' \in N$ such that $\varrho(x_i, x_i') < \varepsilon$. Since $x_{i+1} \in F^\varepsilon(x_i) = \bar B_\varepsilon(F(x_i))$, we can take a $z_{i+1} \in F(x_i)$ such that $\varrho(z_{i+1}, x_{i+1}) < \varepsilon$. We have $\sigma_{z_i} \cap \sigma_{x_i} \cap \sigma_{x_i'} \neq \emptyset$, because $\varrho(z_i, x_i) < \varepsilon$ and $\varrho(x_i, x_i') < \varepsilon$. Since F has cubical values we get $\sigma_{z_{i+1}} \subset F(x_i)$. For each $i \in \mathbb{Z}$ choose a $u_i \in \sigma_{z_i} \cap \sigma_{x_i} \cap \sigma_{x_i'}$. By (5) we get $u_{i+1} \in \sigma_{z_{i+1}} \subset F(x_i) \subset F(u_i)$. Since N is cubical and $x_i' \in N$, we get $u_i \in \sigma_{x_i'} \subset N$. Thus, $u_i \in \operatorname{Inv}(N, F)$ and since $\varrho(x_i, u_i) \le \varrho(x_i, x_i') \le \varepsilon$, we get $x_i \in \bar B_\varepsilon(\operatorname{Inv}(N, F))$. This proves (6).

Now, since $\operatorname{Inv}(N, F)$ as an intersection of cubical sets is cubical and $\operatorname{Inv}(N, F) \subset \operatorname{int} P_1$, we have

$$\bar B_\varepsilon(\operatorname{Inv}(N, F)) \subset \operatorname{int} P_1 \subset P_1 \subset \operatorname{int} \bar B_\varepsilon(P_1).$$

And, since $\operatorname{Inv}(N, F) \cap P_2 = \emptyset$ and the sets are compact, we have $\bar B_\varepsilon(\operatorname{Inv}(N, F)) \cap \bar B_\varepsilon(P_2) = \emptyset$. Hence, $\operatorname{Inv}(\bar B_\varepsilon(N), F^\varepsilon) \subset \operatorname{int}(\bar B_\varepsilon(P_1) \setminus \bar B_\varepsilon(P_2))$, which proves property (c).

In order to prove property (d) it suffices to show that

(7) $$\bar B_\varepsilon(P_1) \setminus \bar B_\varepsilon(P_2) \subset N,$$

because $N \subset \operatorname{int} \bar B_\varepsilon(N)$. Thus, assume that (7) is not true and take an $x \in (\bar B_\varepsilon(P_1) \setminus \bar B_\varepsilon(P_2)) \setminus N$ and choose an $x' \in P_1$ such that $\varrho(x, x') < \varepsilon$. Then $x' \notin P_2$. Let $x'' \in \sigma_x \cap \sigma_{x'}$. Since P is cubical, we have $x'' \in \sigma_{x'} \subset P_1$. We cannot have $x'' \in P_2$, because then $\varrho(x, x'') < \varepsilon$ implies $x \in \bar B_\varepsilon(P_2)$. Therefore, $x'' \in P_1 \setminus P_2 \subset \operatorname{int} N$ by property (d) applied to F and N. We have $\mathring{\sigma}_x \cap N = \emptyset$, because $x \notin N$ and N is cubical. Thus, $x'' \in N \cap \operatorname{cl} \mathring{\sigma}_x \subset \operatorname{bd} N$, a contradiction. ∎

For the sake of simplicity in the next theorem for $A \subset X$ we put $A^\varepsilon := \bar B_\varepsilon(A)$ and $E(A) := X \setminus \operatorname{int} A$.

Theorem 5.12. *Let $F, G : X \rightrightarrows X$ be acyclic upper semicontinuous multivalued maps such that $F \subset G$. Assume that $N \subset X$ is a cubical isolating neighborhood with respect to F, P is a cubical weak index pair in N, N^{ε} is an isolating neighborhood with respect to G, and P^{ε} is a weak index pair for G in N^{ε}. Then the diagram*

$$\begin{tikzcd}
H^*(P_1, P_2) & H^*(P_1 \cup E(N), P_2 \cup E(N)) \arrow[l, "F^*"'] \arrow[r, "\iota_P^*"] & H^*(P_1, P_2) \\
& H^*(P_1^{\varepsilon} \cup E(N), P_2^{\varepsilon} \cup E(N)) \arrow[u, "\lambda^*"'] \arrow[d, "\kappa^*"] & \\
H^*(P_1^{\varepsilon}, P_2^{\varepsilon}) \arrow[uu, "\alpha^*"] & H^*(P_1^{\varepsilon} \cup E(N^{\varepsilon}), P_2^{\varepsilon} \cup E(N^{\varepsilon})) \arrow[l, "G^*"'] \arrow[r, "\iota_{P^{\varepsilon}}^*"] & H^*(P_1^{\varepsilon}, P_2^{\varepsilon}) \arrow[uu, "\alpha^*"']
\end{tikzcd}$$

commutes and $\alpha^*, \kappa^*, \lambda^*$ are isomorphisms for $0 < \varepsilon < \frac{1}{2}\delta$.

Proof. Consider the following diagram

$$\begin{tikzcd}
(P_1, P_2) \arrow[r, "F"] \arrow[dd, hook, "\alpha"] & (P_1 \cup E(N), P_2 \cup E(N)) \arrow[d, hook, "\lambda"] & (P_1, P_2) \arrow[l, hook', "\iota_P"'] \arrow[dd, hook, "\alpha"] \\
& (P_1^{\varepsilon} \cup E(N), P_2^{\varepsilon} \cup E(N)) & \\
(P_1^{\varepsilon}, P_2^{\varepsilon}) \arrow[r, "G"] & (P_1^{\varepsilon} \cup E(N^{\varepsilon}), P_2^{\varepsilon} \cup E(N^{\varepsilon})) \arrow[u, hook, "\kappa"] & (P_1^{\varepsilon}, P_2^{\varepsilon}). \arrow[l, hook', "\iota_{P^{\varepsilon}}"']
\end{tikzcd}$$

The above diagram commutes up to inclusion, that is, $\lambda \circ F \subset \kappa \circ G \circ \alpha$ and $\lambda \circ \iota_P = \kappa \circ \iota_{P^{\varepsilon}} \circ \alpha$. Inclusions $\iota_P, \iota_{P^{\varepsilon}}, \kappa$ induce isomorphisms in cohomology by excision.

Let $\alpha|_{P_i}$ and $\lambda|_{P_i \cup E(N)}$ be restrictions of α, λ to appropriate sets, respectivelyBy Lemma 5.2, inclusions $\alpha|_{P_i} : P_i \hookrightarrow P_i^{\varepsilon}$ and $\lambda|_{P_i \cup E(N)} : P_i \cup E(N) \hookrightarrow P_i^{\varepsilon} \cup E(N)$ induce isomorphisms in cohomology for $i = 1, 2$. Since the following diagram

$$\begin{tikzcd}
P_2 \arrow[r, hook] \arrow[d, hook, "\alpha|_{P_2}"] & P_1 \arrow[r, hook] \arrow[d, hook, "\alpha|_{P_1}"] & (P_1, P_2) \arrow[d, hook, "\alpha"] \\
P_2^{\varepsilon} \arrow[r, hook] & P_1^{\varepsilon} \arrow[r, hook] & (P_1^{\varepsilon}, P_2^{\varepsilon})
\end{tikzcd}$$

commutes, the diagram

$$\begin{tikzcd}
\cdots & H^q(P_2) \arrow[l] & H^q(P_1) \arrow[l] & H^q(P_1, P_2) \arrow[l] & H^{q-1}(P_2) \arrow[l] & \cdots \arrow[l] \\
\cdots & H^q(P_2^{\varepsilon}) \arrow[l] \arrow[u, "(\alpha|_{P_2})^q"] & H^q(P_1^{\varepsilon}) \arrow[l] \arrow[u, "(\alpha|_{P_1})^q"] & H^q(P_1^{\varepsilon}, P_2^{\varepsilon}) \arrow[l] \arrow[u, "\alpha^q"] & H^{q-1}(P_2^{\varepsilon}) \arrow[l] \arrow[u, "(\alpha|_{P_2})^{q-1}"] & \cdots \arrow[l]
\end{tikzcd}$$

also commutes. By Five Lemma, α^* is an isomorphism. An analogous argument for pairs $(P_1 \cup E(N), P_2 \cup E(N))$ and $(P_1^{\varepsilon} \cup E(N), P_2^{\varepsilon} \cup E(N))$ proves that λ^* is an isomorphism too. ∎

Theorem 5.13. *Let $F : X \multimap X$ be a cubical, upper semicontinuous multivalued map with contractible values. Assume that $N \subset X$ is a cubical isolating neighborhood with respect to F, P is a cubical weak index pair in N, and $0 < \varepsilon < \frac{1}{2}\delta$. Then $a_\varepsilon(F) \neq \emptyset$, and every ε-approximation of F has N as an isolating neighborhood and $R := \bar{B}_\varepsilon(P) \cap N$ as a weak index pair. Moreover, index maps I_{F_P} and I_{f_R} are conjugate.*

Proof. By Propositions 5.6 and 5.7, N is an isolating neighborhood for F_ε and P is a weak index pair for F_ε in N. By Lemma 5.4, F_ε is upper semicontinuous and has contractible values. Moreover, $F \subset F_\varepsilon$, showing that index maps I_{F_P} and $I_{F_{\varepsilon P}}$ are conjugate. By Lemma 5.8 and Theorem 5.11 applied for F_ε we infer that $\bar{B}_\varepsilon(N)$ is an isolating neighborhood for $(F_\varepsilon)^\varepsilon$ and $\bar{B}_\varepsilon(P)$ is a weak index pair for $(F_\varepsilon)^\varepsilon$ in $\bar{B}_\varepsilon(N)$. Note that, by Lemma 5.5, $(F_\varepsilon)^\varepsilon$ is upper semicontinuous and has contractible values. Therefore, Theorem 5.12 applied for maps F_ε and $(F_\varepsilon)^\varepsilon$, implies that index maps $I_{F_{\varepsilon P}}$ and $I_{(F_\varepsilon)^\varepsilon{}_{\bar{B}_\varepsilon(P)}}$ are conjugate.

By Proposition 5.9, N is an isolating neighborhood for $(F_\varepsilon)^\varepsilon$. Hence, by [2, Lemma 5.1], R is a weak index pair for $(F_\varepsilon)^\varepsilon$ in N. The diagram

$$\begin{tikzcd}
(R_1, R_2) \arrow[r, "(F_\varepsilon)^\varepsilon"] \arrow[d, hook, "\alpha"] & (R_1 \cup E(N), R_2 \cup E(N)) \arrow[d, hook, "\mathrm{id}"] & (R_1, R_2) \arrow[l, hook', "\iota_R"'] \arrow[d, hook, "\alpha"] \\
(P_1^\varepsilon, P_2^\varepsilon) \arrow[r, "(F_\varepsilon)^\varepsilon"] & (P_1^\varepsilon \cup E(N), P_2^\varepsilon \cup E(N)) & (P_1^\varepsilon, P_2^\varepsilon) \arrow[l, hook', "\iota_{P^\varepsilon}"']
\end{tikzcd}$$

in which inclusions α, ι_R, and ι_{P^ε} are excisions, commutes. This, along with the fact that pairs (R_1, R_2) and $(P_1^\varepsilon \cup E(N), P_2^\varepsilon \cup E(N))$ are associate, shows that index maps $I_{(F_\varepsilon)^\varepsilon{}_{\bar{B}_\varepsilon(P)}}$ and $I_{(F_\varepsilon)^\varepsilon{}_R}$ are conjugate.

Eventually we infer that I_{F_P} and $I_{(F_\varepsilon)^\varepsilon{}_R}$ are conjugate.

The existence of an ε-approximation $f : X \to X$ of F follows from [15, Theorem 5.12] (cf., e.g., [14, Theorem 23.8]). Observe that for an arbitrary ε-approximation $f : X \to X$ of F the inclusion $f \subset (F_\varepsilon)^\varepsilon$ holds. Therefore, index maps I_{f_R} and $I_{(F_\varepsilon)^\varepsilon{}_R}$ are conjugate, and the conclusion follows. ∎

6. Index map and its iterates.

Throughout this section we assume that X is a locally compact metrizable space and $f : X \to X$ is a discrete dynamical system.

For convenience we shall use the notion of associated pairs introduced in [33]. Namely, we say that a pair of paracompact sets $P^0 = (P_1^0, P_2^0)$ is *associated* with a weak index pair P with respect to f, if

(a1) $P \subset P^0$;
(a2) $P_1 \setminus P_2 = P_1^0 \setminus P_2^0$;
(a3) $f(P) \subset P^0$.

Note that if P^0 is associated with a weak index pair P then the pair of pairs (P, P^0) is a weak index quadruple in the sense of [24]. Moreover, by (a2) the inclusion i_{PP^0} induces an isomorphism in the Alexander–Spanier cohomology, and by (a3), we can consider the restriction f_{PP^0} of f to the domain of P as a map of pairs $f_{PP^0} : P \to P^0$.

Clearly, the pair $T_N(P)$ is associated with P. Another pair associated with P is

$$S_N(P) := (S_{N,1}(P), S_{N,2}(P)) := (P_1 \cup (f(P_1) \setminus \operatorname{int} N), P_2 \cup (f(P_1) \setminus \operatorname{int} N)).$$

Observe that $S_N(P)$ is the smallest pair associated with P, i.e., for any pair P' associated with P, we have $S_{N,i}(P) \subset P'_i$. Indeed, for $i = 1$ the inclusion follows directly from (a1) and (a3). Note that in order to show the inclusion $S_{N,2}(P) = P_2 \cup (f(P_1) \setminus \operatorname{int} N) \subset P'_2$ it suffices to verify that $f(P_1) \setminus \operatorname{int} N \subset P'_2$, as $P_2 \subset P'_2$ by (a1). Suppose to the contrary that there exists a $y \in (f(P_1) \setminus \operatorname{int} N) \setminus P'_2$. Then, by (a3) and (a2), $y \in P'_1 \setminus P'_2 = P_1 \setminus P_2$. However, $P_1 \setminus P_2 \subset \operatorname{int} N$; hence $y \in \operatorname{int} N$, a contradiction.

We have the commutative diagram

$$\begin{tikzcd}
 & & (T_{N,1}(P), T_{N,2}(P)) & & \\
(P_1, P_2) \arrow[urr, "f_P"] \arrow[rr, "f_{PS(P)}"] \arrow[drr, "f_{PP'}"'] & & (S_{N,1}(P), S_{N,2}(P)) \arrow[u, "j_1"'] \arrow[d, "j_2"] & & (P_1, P_2) \arrow[ull, "i_P"'] \arrow[ll, "i_{PS(P)}"'] \arrow[dll, "i_{PP'}"] \\
 & & (P'_1, P'_2) & &
\end{tikzcd}$$

in which i_P, $i_{PP'}$, $i_{PS(P)}$, j_1, and j_2 are inclusions. Since any of the pairs in the diagram is associated with P, each of the inclusions induces an isomorphism in cohomology. Hence, by the commutativity of the diagram we obtain $I_{f_P} = f^*_{PP'} \circ (i^*_{PP'})^{-1}$. For reference we state this observation as the following.

Proposition 6.1. *Let P be a weak index pair for f and let P' be a pair associated with P. Then*

(i) *there is a well-defined map of pairs $f_{PP'} : P \ni x \mapsto f(x) \in P'$;*
(ii) *the inclusion $i_{PP'} : P \to P'$ induces an isomorphism in cohomology;*
(iii) $I_{f_P} = f^*_{PP'} \circ (i^*_{PP'})^{-1}$.

Proposition 6.2. *Let M be an isolating neighborhood for f. For any $n \in \mathbb{N}$ there exists an open neighborhood U of $\operatorname{Inv}(M, f)$ with $\operatorname{cl} U \subset M$ such that for any $x \in U$ we have*

$$f^k(x) \in \operatorname{int} M \text{ for } k \in I_n.$$

Proof. Since S is compact and f is continuous, we can find an open set $U \supset S$ with $\operatorname{cl} U \subset M$ such that $f(U) \cup f^2(U) \cup \cdots \cup f^k(U) \subset \operatorname{int} M$. ■

The following proposition is straightforward.

Proposition 6.3. *If N is an isolating neighborhood for f then for any $k \in \mathbb{N}$ we have*

$$\operatorname{Inv}(N, f) \subset \operatorname{Inv}(N, f^k). \tag{8}$$

Although the converse inclusion is not valid in general, we have the following proposition.

Proposition 6.4. *Let S be an isolated invariant set with respect to f. For any $k \in \mathbb{N}$ there exists an isolating neighborhood M of S such that*

$$\operatorname{Inv}(M, f) = S = \operatorname{Inv}(M, f^k). \tag{9}$$

Proof. Let $\hat{N}$ be an isolating neighborhood of S with respect to f. By Proposition 6.2 we can take an open neighborhood U of S such that $\bigcup_{i=1}^{k} f^k(U) \subset \operatorname{int} \hat{N}$. Let $M \subset U$ be an isolating neighborhood of S. We have $S = \operatorname{Inv}(M, f) \subset \operatorname{Inv}(M, f^k)$. To see the opposite inclusion take an $x \in \operatorname{Inv}(M, f^k)$. Then $f^{ik}(x) \in \operatorname{Inv}(M, f^k)$ for $i \in \mathbb{Z}$. But $\operatorname{Inv}(M, f^k) \subset M \subset U$, therefore $f^j(f^{ik}(x)) \in \operatorname{int} \hat{N}$ for $j \in I_k$. Hence, $x \in \operatorname{Inv}(\hat{N}, f) = S$. ■

Proposition 6.5. *Let S be an isolated invariant set for f. For any $n \in \mathbb{N}$ there exist isolating neighborhoods $N \subset M$ of S and weak index pairs P and Q, respectively, in N and M, such that for each $k \in I_n$*

(i) *P is a weak index pair for S and f^k;*
(ii) *Q is associated with P with respect to f^k;*
(iii) *$T_N(P)$ is associated with Q with respect to f.*

Proof. Fix an arbitrary $n \in \mathbb{N}$ and consider an isolating neighborhood M of S satisfying (9). Take $U \subset \operatorname{cl} U \subset M$, an open neighborhood of S as in Proposition 6.2, and a compact set $N \subset U$ with $S \subset \operatorname{int} N$. Note that such an N is an isolating neighborhood for f^k for each $k \in \{1, 2, \ldots, n\}$. By [2, Theorem 4.12] we can find a weak index pair $Q = (Q_1, Q_2)$ for f and S in M such that $Q_1 \setminus Q_2 \subset \operatorname{int} N$. Define the pair $P := (P_1, P_2)$ as the intersection

$$P := Q \cap N. \tag{10}$$

According to [2, Lemma 5.1], P is a weak index pair for f in N. We shall prove that the pairs P and Q satisfy assertions (i), (ii), and (iii).

First we prove that

$$f^k(P) \subset Q \text{ for } k \in I_n. \tag{11}$$

We argue by induction with respect to k. Since for $i = 1, 2$ we have $P_i \subset N \subset U$, by Proposition 6.2, we get $f(P_i) \subset M$. Therefore, $f(P_i) \subset f(Q_i) \cap M \subset Q_i$, as $P_i \subset Q_i$ and Q_i is positively invariant with respect to f and M. Next, suppose that for some $k \in I_{n-1}$ we have $f^k(P_i) \subset Q_i$. By Proposition 6.2, $f^{k+1}(P_i) \subset M$. Consequently, $f^{k+1}(P_i) \subset f(f^k(P_i)) \cap M \subset f(Q_i) \cap M \subset Q_i$. This completes the proof of (11).

We shall prove that P is a weak index pair with respect to each f^k, $k \in I_n$. To this end fix an arbitrary $k \in \{2, \ldots, n\}$ (recall that for $k = 1$ the assertion follows from [2, Lemma 5.1]). Since P is a weak index pair in $N \subset U$ with respect to f, we have $P_1 \setminus P_2 \subset \operatorname{int} N$, as well as $\operatorname{Inv}(N, f^k) = \operatorname{Inv}(N, f) \subset \operatorname{int}(P_1 \setminus P_2)$. This shows that P satisfies properties (c) and (d) of Definition 2.1 of a weak index pair for f^k. Since property (a) follows easily from (11), it remains to verify property (b), that is, $\operatorname{bd}_{f^k}(P_1) \subset P_2$. Suppose to the contrary that there exists a $y \in \operatorname{bd}_{f^k}(P_1) \setminus P_2$. Then $y \in P_1 \setminus P_2$ and $y \in \operatorname{cl}(f^k(P_1) \setminus P_1)$. Consider a sequence $\{y_n\} \subset f^k(P_1) \setminus P_1$ convergent to y. Since $y \in P_1 \setminus P_2 \subset \operatorname{int} N$, for sufficiently large n we have $y_n \in \operatorname{int} N$. Consequently, $y_n \in f^k(P_1) \cap \operatorname{int} N \subset f^k(P_1) \cap N$, which according to the property (a) of P yields $y_n \in P_1$, a contradiction.

To prove (ii) observe that properties (a1) and (a2) are obvious and (a3) follows from (11).

We shall show (iii). Since $N \subset M$, by (10) it follows that $Q \subset T_N(P)$, showing that (a1) is satisfied. Condition (a2) is a direct consequence of (ii) and the fact that $T_N(P)$ is associated with P. It remains to verify property (a3). By (10) and the inclusion $N \subset M$ it follows that $T_M(Q) \subset T_N(P)$. This, along with the obvious inclusion $f(Q) \subset T_M(Q)$, implies $f(Q) \subset T_N(P)$, and completes the proof. ■

Proposition 6.6. *Let $N \subset M$ be isolating neighborhoods of S. Assume that P is a weak index pair in N with respect to each f^k, $k \in I_p$, and Q is a weak index pair with respect to f in M. Moreover, assume that Q is associated with P with respect to f^k, and $T_N(P)$ is associated with Q with respect to f. Then*

$$I_{f_P^p} = I_{f_P}^p. \tag{12}$$

Proof. Fix an arbitrary $k \in I_p$. Since Q is associated with P with respect to f^k, by Proposition 6.1 we have

$$I_{f_P^k} = (f_{PQ}^k)^* \circ (i_{PQ}^*)^{-1}. \tag{13}$$

Note that, for each $k \in I_{p-1}$, we have the commutative diagram

$$\begin{tikzcd}
(Q_1, Q_2) \arrow[drr, "f_{QT_N(P)}"] & & (P_1, P_2) \arrow[ll, "i_{PQ}"'] \arrow[d, "f_P"] & & \\
(P_1, P_2) \arrow[u, "f_{PQ}^k"] \arrow[rr, "f_P^{k+1}"] & & (T_{N,1}(P), T_{N,2}(P)) & & (P_1, P_2) \arrow[ll, "i_P"']
\end{tikzcd}$$

in which i_P and i_{PQ} are inclusions. Moreover, the inclusions i_P and i_{PQ} induce isomorphisms in cohomology, as excisions. Therefore, by the commutativity of the diagram and (13) we obtain

$$\begin{aligned}
I_{f_P^{k+1}} &= (f_P^{k+1})^* \circ (i_P^*)^{-1} \\
&= (f_{PQ}^k)^* \circ (i_{PQ}^*)^{-1} \circ (f_P)^* \circ (i_P^*)^{-1} \\
&= I_{f_P^k} \circ I_{f_P}.
\end{aligned}$$

Taking into account that the above equality is valid for an arbitrary $k \in I_{p-1}$, the assertion follows by induction. ■

Proposition 6.7. *Assume N is an isolating neighborhood with respect to f and P is a weak index pair for f in N. Moreover, assume $N = \bigcup_{i=1}^n N_i$, where N_i are pairwise disjoint compact subsets of N. Then, for any $I \subset I_n$, the union $\bigcup_{i \in I} N_i =: N_I$ is an isolating neighborhood for f, and $Q := P \cap N_I$ is a weak index pair for f in N_I.*

Proof. Clearly, N_I is compact. Since $\operatorname{int} N_I = N_I \cap \operatorname{int} N$ and N is an isolating neighborhood for f, we have the inclusions $\operatorname{Inv}(N_I, f) \subset \operatorname{Inv}(N, f) \cap N_I \subset \operatorname{int} N \cap N_I = \operatorname{int} N_I$, showing that N_I is an isolating neighborhood for f.

We shall verify that Q is a weak index pair in N_I. It is obvious that $Q_2 \subset Q_1$ are compact subsets of N_I. For the proof of condition (a) in Definition 2.1 observe that $f(Q_i) \cap N_I \subset f(P_i) \cap N \subset P_i$, hence $f(Q_i) \cap N_I \subset P_i \cap N_I = Q_i$. Moreover, we have the inclusions

$\operatorname{Inv}(N_I, f) \subset \operatorname{int} N_I \cap \operatorname{Inv}(N, f) \subset \operatorname{int} N_I \cap \operatorname{int}(P_1 \setminus P_2) = \operatorname{int}(Q_1 \setminus Q_2)$, showing that Q satisfies condition (c). Next, observe that $Q_1 \setminus Q_2 = (P_1 \setminus P_2) \cap N_I \subset \operatorname{int} N \cap N_I = \operatorname{int} N_I$, which means that Q satisfies condition (d). We still need to show that Q satisfies property (b). Suppose to the contrary that there exists a $y \in \operatorname{bd}_{f}(Q_1) \setminus Q_2$. Then $y \in Q_1 \setminus Q_2$ and $y \in \operatorname{cl}(f(Q_1) \setminus Q_1)$. Thus we can take a sequence $\{y_n\} \subset f(Q_1) \setminus Q_1$ convergent to y. By the inclusion $y \in Q_1 \setminus Q_2 \subset \operatorname{int} N_I$, it follows that $y_n \in \operatorname{int} N_I$ for sufficiently large n. Consequently, $y_n \in f(Q_1) \cap \operatorname{int} N_I \subset f(Q_1) \cap N_I$, which according to the positive invariance of Q_1 with respect to f and N_I yields $y_n \in Q_1$, a contradiction. ■

Proposition 6.8. *Assume that N is an isolating neighborhood for f and P is a weak index pair in N. Moreover, assume $N = N_1 \cup N_2$, where N_1, N_2 are compact disjoint subsets of N. Let $P^1 := P \cap N_1$, let $\iota : H^*(P^1) \to H^*(P^1) \times H^*(P^2)$ be the inclusion, and let $\pi : H^*(P^1) \times H^*(P^2) \to H^*(P^1)$ be the projection. Then*

$$I_{f_{P^1}} = \pi \circ I_{f_P} \circ \iota. \tag{14}$$

Proof. By Proposition 6.7, N_1 is an isolating neighborhood for f, and P^1 is a weak index pair in N^1. Therefore, we have well-defined index maps $I_{f_{P^1}}$ and I_{f_P}, associated with the weak index pairs P^1 and P, respectively.

Consider the commutative diagram

$$\begin{tikzcd}
(P_1, P_2) \arrow[r, "f_P"] & (T_{N,1}(P), T_{N,2}(P)) \arrow[d, "k"] & (P_1, P_2) \arrow[l, "i_P"'] \\
(P_1^1, P_2^1) \arrow[u, "j"] \arrow[r, "f_{P^1}"] & (T_{N^1,1}(P^1), T_{N^1,2}(P^1)) & (P_1^1, P_2^1) \arrow[l, "i_{P^1}"'] \arrow[u, "j"']
\end{tikzcd}$$

in which i_P, i_{P^1}, j, and k are inclusions. Recall that i_P and i_{P^1} induce isomorphisms in cohomology by the strong excision property. By the commutativity of the diagram we obtain $j^* \circ f_P^* \circ (i_P^*)^{-1} = f_{P^1}^* \circ (i_{P^1}^*)^{-1} \circ j^*$, showing that

$$j^* \circ I_{f_P} = I_{f_{P^1}} \circ j^*. \tag{15}$$

Consider the commutative diagram

$$\begin{tikzcd}
(P_1, P_2) \arrow[dr, "\kappa"] & \\
(P_1^1, P_2^1) \arrow[u, "j"] \arrow[r, "\lambda"] & (P_1, P_2 \cup P_1^2)
\end{tikzcd}$$

in which κ and λ are inclusions. Note that λ induces an isomorphism in cohomology as an excision. Moreover, by the commutativity of the diagram, $\lambda^* = j^* \circ \kappa^*$, showing that

$$j^* \circ (\kappa^* \circ (\lambda^*)^{-1}) = \operatorname{id}_{H^*(P_1^1, P_2^1)}. \tag{16}$$

Note that $\kappa^* \circ (\lambda^*)^{-1} : H^*(P^1) \to H^*(P^1) \times H^*(P^2)$ is an inclusion. Thus, $\kappa^* \circ (\lambda^*)^{-1} = \iota$ and $j^* = \pi$. Now, (14) follows from (15), which completes the proof. ■

7. Determining orbits via the Ważewski property of the Conley index. Let X be a locally compact metrizable space, and let $f : X \to X$ be a discrete dynamical system. Recall that for $p \geq 2$ we denote by $\mathbb{Z}_p := \{0, 1, \dots, p-1\}$ the topological group with the addition modulo p and the discrete topology. We define the space $\bar{X} := X \times \mathbb{Z}_p$ with the product topology, and dynamical systems $\bar{f}, \underline{f} : \bar{X} \to \bar{X}$, by

(17) $$\bar{f} : \bar{X} \ni (x, i) \longmapsto (f(x), i + 1) \in \bar{X}$$

and

$$\underline{f} : \bar{X} \ni (x, i) \longmapsto (f(x), i) \in \bar{X},$$

respectively. Consider the homeomorphism

$$l : \bar{X} \ni (x, i) \longmapsto (x, i + 1) \in \bar{X}$$

and observe that we have

(18) $$\bar{f} = \underline{f} \circ l = l \circ \underline{f}.$$

Given $A \subset X$, by $\bar{A}$ we shall denote the set $A \times \mathbb{Z}_p$.

Proposition 7.1. *If N is an isolating neighborhood for f then $\bar{N}$ is an isolating neighborhood for both $\bar{f}$ and $\underline{f}$. Moreover, if P is a weak index pair for f in N then $\bar{P}$ is a weak index pair in $\bar{N}$ for both $\bar{f}$ and $\underline{f}$.*

Proof. Consider the dynamical system $\bar{f}$. Clearly $\bar{N}$ is compact. We shall verify that $\mathrm{Inv}(\bar{N}, \bar{f}) \subset \mathrm{int}\, \bar{N}$. To this end consider $\bar{x} = (x, i) \in \mathrm{Inv}(\bar{N}, \bar{f})$. Let $\bar{\sigma} : \mathbb{Z} \to \bar{X}$ be a solution for $\bar{f}$ passing through $\bar{x}$, which is contained in $\bar{N}$, that is, $\bar{\sigma}(0) = \bar{x}$, $\bar{\sigma}(\mathbb{Z}) \subset \bar{N}$, and $\bar{\sigma}(k + 1) = \bar{f}(\bar{\sigma}(k))$ for $k \in \mathbb{Z}$. Define $\sigma : \mathbb{Z} \to X$ by $\sigma(k) := p(\bar{\sigma}(k))$ for $k \in \mathbb{Z}$, where $p : \bar{X} \ni (x, i) \mapsto x \in X$ denotes the projection. One can easily see that σ is a solution for f through x in N. Therefore, $x \in \mathrm{int}\, N$, as N is an isolating neighborhood for f. This shows that $\bar{x} = (x, i) \in \mathrm{int}\, N \times \mathbb{Z}_p = \mathrm{int}\, \bar{N}$, and completes the proof.

The verification that $\bar{P}$ is a weak index pair for $\bar{f}$ and $\bar{N}$ is straightforward.

The proof for $\underline{f}$ is similar. ■

For $i \in \mathbb{Z}_p$ define the map

(19) $$\mu_i : X \ni x \mapsto (x, i) \in X \times \{i\}.$$

The following proposition is straightforward.

Proposition 7.2. *Assume that N is an isolating neighborhood for f, and P is a weak index pair in N. For any $i \in \mathbb{Z}_p$ the set $N \times \{i\}$ is an isolating neighborhood for $\underline{f}$, and $P \times \{i\}$ is a weak index pair in $N \times \{i\}$. Moreover,*

$$I_{f_P} \circ \mu_i^* = \mu_i^* \circ I_{\underline{f}_{P \times \{i\}}}.$$

Proposition 7.3. *Assume that N is an isolating neighborhood for f, and P is a weak index pair in N. We have*

$$\left(\times_{i=0}^{p-1} \mu_i^*\right) \circ I_{\bar{f}_{\bar{P}}} = \left(\times_{i=0}^{p-1} I_{f_P}\right) \circ \left(\times_{i=0}^{p-1} \mu_{i+1}^*\right),$$

where $\times_{i=0}^{p-1} \mu_i^* : H^*(\bar{X}) \to \times_{i=0}^{p-1} H^*(X)$.

Proof. By Proposition 7.1 the pair $\bar P$ is a weak index pair with respect to $\bar f$. Therefore, the restriction $\bar f_{\bar P}$ of $\bar f$ to the domain $\bar P$ is a map of pairs

$$\bar f_{\bar P} : \bar P \to T_{\bar N}(\bar P).$$

We claim that

$$I_{\bar f_{\bar P}} = I_{f_{\bar P}} \circ l^*. \tag{20}$$

Indeed, note that $l \circ i_{\bar P} = i_{\bar P} \circ l$. Hence, $l^* \circ (i^*_{\bar P})^{-1} = (i^*_{\bar P})^{-1} \circ l^*$ and, by the second equality in (18), we get

$$\begin{aligned} I_{\bar f_{\bar P}} &= \bar f^*_{\bar P} \circ (i^*_{\bar P})^{-1} \\ &= (l \circ f_{\bar P})^* \circ (i^*_{\bar P})^{-1} \\ &= f^*_{\bar P} \circ l^* \circ (i^*_{\bar P})^{-1} \\ &= f^*_{\bar P} \circ (i^*_{\bar P})^{-1} \circ l^* \\ &= I_{f_{\bar P}} \circ l^*. \end{aligned}$$

For $i \in \mathbb{Z}_p$ denote by l the restriction of l to the domain $X \times \{i\}$, and observe that $l_i \circ \mu_i = \mu_{i+1}$. Hence, $\mu^*_i \circ l^*_i = \mu^*_{i+1}$ and we have

$$(\times_{i=0}^{p-1} \mu^*_i) \circ l^* = (\times_{i=0}^{p-1} \mu^*_i) \circ (\times_{i=0}^{p-1} l^*_i) = \times_{i=0}^{p-1} (\mu^*_i \circ l^*_i) = \times_{i=0}^{p-1} \mu^*_{i+1}. \tag{21}$$

Therefore, according to (20), in order to complete the proof it suffices to verify that

$$(\times_{i=0}^{p-1} \mu^*_i) \circ I_{f_{\bar P}} = (\times_{i=0}^{p-1} I_{f_P}) \circ (\times_{i=0}^{p-1} \mu^*_i).$$

Since $\bar P$ is a union of pairwise disjoint sets $P \times \{i\}$, we have the product decomposition of $H^*(\bar P) = \times_{i=0}^{p-1} H^*(P \times \{i\})$. Similarly, $H^*(T_{\bar N}(\bar P)) = \times_{i=0}^{p-1} H^*(T_N(P) \times \{i\})$, as the sets $T_N(P) \times \{i\}$ are pairwise disjoint. According to the definition of $\bar f$ and Proposition 7.2, we can consider the restriction $f_{P \times \{i\}}$ of $\bar f$ to the domain $P \times \{i\}$ as a map of pairs

$$f_{P \times \{i\}} : P \times \{i\} \to T_N(P) \times \{i\}.$$

Thus, we have

$$f^*_{\bar P} = \times_{i=0}^{p-1} f^*_{P \times \{i\}}.$$

Similarly,

$$i^*_{\bar P} = \times_{i=0}^{p-1} i^*_{P \times \{i\}}.$$

Consequently,

$$\begin{aligned} I_{f_{\bar P}} &= f^*_{\bar P} \circ \left(i^*_{\bar P}\right)^{-1} \\ &= \left(\times_{i=0}^{p-1} f^*_{P \times \{i\}}\right) \circ \left(\times_{i=0}^{p-1} i^*_{P \times \{i\}}\right)^{-1} \\ &= \times_{i=0}^{p-1} \left(f^*_{P \times \{i\}} \circ \left(i^*_{P \times \{i\}}\right)^{-1}\right) \\ &= \times_{i=0}^{p-1} I_{f_{P \times \{i\}}}. \end{aligned}$$

Now, by Proposition 7.2, we obtain

$$\begin{aligned}\left(\times_{i=0}^{p-1}\mu_i^*\right)\circ I_{\bar f_{\bar P}} &= \left(\times_{i=0}^{p-1}\mu_i^*\right)\circ\left(\times_{i=0}^{p-1} I_{\bar f_{P\times\{i\}}}\right)\\ &= \times_{i=0}^{p-1}\left(\mu_i^*\circ I_{\bar f_{P\times\{i\}}}\right)\\ &= \times_{i=0}^{p-1}\left(I_{f_P}\circ\mu_i^*\right)\\ &= \left(\times_{i=0}^{p-1} I_{f_P}\right)\circ\left(\times_{i=0}^{p-1}\mu_i^*\right),\end{aligned}$$

which completes the proof. ■

From now on we assume that $N=\bigcup_{i=1}^n N_i$, where N_i are pairwise disjoint compact subsets of N, N is an isolating neighborhood with respect to f, and P is a weak index pair for f in N. Denote $P^i := P\cap N_i$. Let $p\in\mathbb{N}$ and let $\sigma:=(\sigma_0,\dots,\sigma_{p-1})\in I_n^{\mathbb{Z}_p}$.

$$I_\sigma := \times_{i=0}^{p-1}\left(\pi_{\sigma_i}\circ I_{f_P}\circ\iota_{\sigma_{i+1}}\right), \tag{22}$$

where $\pi_i : H^*(P)\to H^*(P^i)$ are projections, and $\iota_i : H^*(P^i)\to H^*(P)$ are inclusions.

Consider the dynamical system $\bar f$ on $\bar X$ given by (17). For $\sigma\in I_n^{\mathbb{Z}_p}$ set

$$N_\sigma := \bigcup_{i=0}^{p-1}(N_{\sigma_i}\times\{i\})$$

and let

$$S_\sigma := \operatorname{Inv}(N_\sigma,\bar f). \tag{23}$$

Proposition 7.4. *The set S_σ is an isolated invariant set for $\bar f$, N_σ is its isolating neighborhood, and there exists a weak index pair R for $\bar f$ and S_σ such that*

$$I_\sigma\circ\left(\times_{i=0}^{p-1}\mu_{i+1}^*\right) = \left(\times_{i=0}^{p-1}\mu_i^*\right)\circ I_{\bar f_R}. \tag{24}$$

Moreover, $I_{\bar f_R}^p$ and I_σ^p are conjugate.

Proof. First note that, by Proposition 7.1, $\bar N$ is an isolating neighborhood for $\bar f$, and $\bar P$ is a weak index pair in $\bar N$. Clearly, N_σ is a compact subset of $\bar N$; hence, according to Proposition 6.7, N_σ is an isolating neighborhood for $\bar f$, and $R := \bar P\cap N_\sigma$ is a weak index pair in N_σ. Therefore, we have a well-defined index map $I_{\bar f_R}$ for $\bar f$, associated with the weak index pair R.

We shall prove that $I_{\bar f_R}$ and I_σ satisfy (24). To this end consider projections

$$\bar\pi_{k,i} : H^*(P\times\{i\})\to H^*(P^k\times\{i\})$$

and the inclusions $\bar\iota_{k,i} : H^*(P^k\times\{i\})\to H^*(P\times\{i\})$ for $k\in I_n$ and $i\in\mathbb{Z}_p$. One can observe that, for any $i \in \mathbb{Z}_p$, we have

$$\mu_i^* \circ \bar{\pi}_{k,i} = \pi_k \circ \mu_i^*$$

and

$$\mu_i^* \circ \bar{\iota}_{k,i} = \iota_k \circ \mu_i^*.$$

Using the above identities and Proposition 7.3 we obtain

$$\begin{aligned}
I_\sigma \circ \left(\times_{i=0}^{p-1} \mu_{i+1}^*\right) &= \left(\times_{i=0}^{p-1} \pi_{\sigma_i}\right) \circ \left(\times_{i=0}^{p-1} I_{f_P}\right) \circ \left(\times_{i=0}^{p-1} \iota_{\sigma_{i+1}}\right) \circ \left(\times_{i=0}^{p-1} \mu_{i+1}^*\right)\\
&= \left(\times_{i=0}^{p-1} \pi_{\sigma_i}\right) \circ \left(\times_{i=0}^{p-1} I_{f_P}\right) \circ \left(\times_{i=0}^{p-1} \iota_{\sigma_{i+1}} \circ \mu_{i+1}^*\right)\\
&= \left(\times_{i=0}^{p-1} \pi_{\sigma_i}\right) \circ \left(\times_{i=0}^{p-1} I_{f_P}\right) \circ \left(\times_{i=0}^{p-1} \mu_{i+1}^* \circ \bar{\iota}_{\sigma_{i+1},i+1}\right)\\
&= \left(\times_{i=0}^{p-1} \pi_{\sigma_i}\right) \circ \left(\times_{i=0}^{p-1} I_{f_P}\right) \circ \left(\times_{i=0}^{p-1} \mu_{i+1}^*\right) \circ \left(\times_{i=0}^{p-1} \bar{\iota}_{\sigma_{i+1},i+1}\right)\\
&= \left(\times_{i=0}^{p-1} \pi_{\sigma_i}\right) \circ \left(\times_{i=0}^{p-1} \mu_i^*\right) \circ I_{\bar{f}_{\bar{P}}} \circ \left(\times_{i=0}^{p-1} \bar{\iota}_{\sigma_{i+1},i+1}\right)\\
&= \left(\times_{i=0}^{p-1} (\pi_{\sigma_i} \circ \mu_i^*)\right) \circ I_{\bar{f}_{\bar{P}}} \circ \left(\times_{i=0}^{p-1} \bar{\iota}_{\sigma_{i+1},i+1}\right)\\
&= \left(\times_{i=0}^{p-1} (\mu_i^* \circ \bar{\pi}_{\sigma_i,i})\right) \circ I_{\bar{f}_{\bar{P}}} \circ \left(\times_{i=0}^{p-1} \bar{\iota}_{\sigma_{i+1},i+1}\right)\\
&= \left(\times_{i=0}^{p-1} \mu_i^*\right) \circ \left(\times_{i=0}^{p-1} \bar{\pi}_{\sigma_i,i}\right) \circ I_{\bar{f}_{\bar{P}}} \circ \left(\times_{i=0}^{p-1} \bar{\iota}_{\sigma_{i+1},i+1}\right).
\end{aligned}$$

Note that $\times_{i=0}^{p-1} \bar{\pi}_{\sigma_i,i}$ is the projection of $H^*(\bar{P})$ onto $H^*(R)$, and $\times_{i=0}^{p-1} \bar{\iota}_{\sigma_{i+1},i+1}$ is the inclusion of $H^*(R)$ into $H^*(\bar{P})$. Hence, applying Proposition 6.8 we get (24).

We shall prove that

$$I_\sigma^p \circ \left(\times_{i=0}^{p-1} \mu_{i+1}^*\right) = \left(\times_{i=0}^{p-1} \mu_{i+1}^*\right) \circ I^p_{\bar{f}_R}. \tag{25}$$

Note that $\bar{f}_R \circ l = l \circ \bar{f}_R$. Hence, $(l^*)^{-1} \circ \bar{f}_R^* = \bar{f}_R^* \circ (l^*)^{-1}$. Similarly, $(l^*)^{-1} \circ (i_R^*)^{-1} = (i_R^*)^{-1} \circ (l^*)^{-1}$, as $i_R \circ l = l \circ i_R$. We have

$$\begin{aligned}
(l^*)^{-1} \circ I_{\bar{f}_R} &= (l^*)^{-1} \circ \bar{f}_R^* \circ (i_R^*)^{-1}\\
&= \bar{f}_R^* \circ (l^*)^{-1} \circ (i_R^*)^{-1}\\
&= \bar{f}_R^* \circ (i_R^*)^{-1} \circ (l^*)^{-1}\\
&= I_{\bar{f}_R} \circ (l^*)^{-1}.
\end{aligned}$$

Therefore, using (24) and (21), we obtain

$$\begin{aligned}
I_\sigma^p \circ \left(\times_{i=0}^{p-1} \mu_{i+1}^*\right) &= I_\sigma^{p-1} \circ \left(\times_{i=0}^{p-1} \mu_i^*\right) \circ I_{\bar{f}_R}\\
&= I_\sigma^{p-1} \circ \left(\times_{i=0}^{p-1} \mu_{i+1}^*\right) \circ (l^*)^{-1} \circ I_{\bar{f}_R}\\
&= I_\sigma^{p-1} \circ \left(\times_{i=0}^{p-1} \mu_{i+1}^*\right) \circ I_{\bar{f}_R} \circ (l^*)^{-1}.
\end{aligned}$$

Now, by the reverse induction with respect to p and the fact that $\left((l^*)^{-1}\right)^p$ is the identity, we get (25). This shows that $I^p_{\bar{f}_R}$ and I^p_σ are conjugate, and completes the proof. ■

We are ready to present the main theorems of this section. They show that from the index map for f, itself, we can extract information which is sufficient to justify the existence of an orbit of f, passing through the components of N in a given order.

Theorem 7.5. *Assume that $N = \bigcup_{i=1}^{n} N_i$, where N_i are pairwise disjoint compact subsets of N, N is an isolating neighborhood with respect to f, and P is a weak index pair for f in N. Let $p \in \mathbb{N}$ and let $\sigma := (\sigma_0, \ldots, \sigma_{p-1}) \in I_n^{\mathbb{Z}_p}$. If the endomorphism I_σ given by* (22) *is not nilpotent then there exists a trajectory $\tau : \mathbb{Z} \to \operatorname{Inv}(\bigcup_{i=0}^{p-1} N_{\sigma_i}, f)$ for f, such that $\tau(i + kp) \in N_{\sigma_i}$, for $i \in I_p$, $k \in \mathbb{Z}$.*

Proof. By Proposition 7.4, $S_\sigma = \operatorname{Inv}(\underline{N}_\sigma, \bar{f})$ is an isolated invariant set for $\bar{f}$. Thus, we have a well-defined Conley index $C(\underline{S}_\sigma, \bar{f})$ for S_σ and $\bar{f}$. Note that, by Proposition 7.4, there exists a weak index pair R in $\bar{X}$ for $\bar{f}$ and S_σ, such that $I_{\bar{f}_R}$ and I_σ satisfy (24). Since I_σ is not nilpotent, then so is $I_{\bar{f}_R}$. Consequently, $C(S_\sigma, \bar{f}) \neq 0$. By the Ważewski property of the Conley index (cf. [25, Proposition 2.10]), it follows that $S_\sigma \neq \emptyset$. According to definition (17) of $\bar{f}$, there exists an $x \in N_{\sigma_0}$ such that $(x, 0) \in S_\sigma$. Let $\eta : \mathbb{Z} \to S_\sigma$ be a trajectory for $\bar{f}$ in S_σ through $(x, 0)$. One easily verifies that then $\tau := p \circ \eta$, where $p : \bar{X} \ni (x, i) \mapsto x \in X$ is the projection, is a trajectory for f satisfying the assertion. ∎

For a given $i \in I_n$ define endomorphism $g_i : H^*(P) \to H^*(P)$ by

$$g_i := I_{f_P} \circ \iota_i \circ \pi_i. \tag{26}$$

We are going to prove the following theorem which may be viewed as a counterpart of Theorem 7.5 expressed in terms of compositions of endomorphisms g_i.

Theorem 7.6. *Assume that $N = \bigcup_{i=1}^{n} N_i$, where N_i are pairwise disjoint compact subsets of N, N is an isolating neighborhood with respect to f, and P is a weak index pair for f in N. Let $p \in \mathbb{N}$, let $\sigma := (\sigma_0, \ldots, \sigma_{p-1}) \in I_n^{\mathbb{Z}_p}$ and let endomorphisms g_i be given by* (26). *If the composition $g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}}$ is not nilpotent then there exists a trajectory $\tau : \mathbb{Z} \to \operatorname{Inv}(\bigcup_{i=0}^{p-1} N_{\sigma_i}, f)$ for f, such that $\tau(i + kp) \in N_{\sigma_i}$, for $i \in I_p$, $k \in \mathbb{Z}$.*

For its proof we need an auxiliary lemma. Consider the projections

$$r_i : \bigtimes_{i=0}^{p-1} H^*(P^{\sigma_i}) \to H^*(P^i)$$

and the inclusions

$$m_i : H^*(P^i) \to \bigtimes_{i=0}^{p-1} H^*(P^{\sigma_i}).$$

Let $h_i : \bigtimes_{i=0}^{p-1} H^*(P^{\sigma_i}) \to \bigtimes_{i=0}^{p-1} H^*(P^{\sigma_i})$ be given by

$$h_i := I_\sigma \circ m_i \circ r_i. \tag{27}$$

Let $\operatorname{Perm}(\mathbb{Z}_p)$ and $\operatorname{Cycle}(\mathbb{Z}_p) \subset \operatorname{Perm}(\mathbb{Z}_p)$ stand for the sets of all permutations and all cyclic translations of $\mathbb{Z}_p$, respectively.

Lemma 7.7. *Assume I_σ, g_i, and h_i are given by (22), (26), and (27), respectively. Then*
(i) $I_\sigma^p = \sum_{s\in \mathrm{Cycle}(\mathbb{Z}_p)} (h_{\sigma_{s(0)}} \circ \cdots \circ h_{\sigma_{s(p-1)}})$;
(ii) $h_{\sigma_0} \circ \cdots \circ h_{\sigma_{p-1}} = m_{\sigma_{p-1}} \circ \pi_{\sigma_{p-1}} \circ g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}} \circ \iota_{\sigma_{p-1}} \circ r_{\sigma_{p-1}}$.

Proof. One can observe that

$$I_\sigma = \sum_{i=0}^{p-1} h_{\sigma_i}. \tag{28}$$

Since $h_{\sigma_j} \circ h_{\sigma_i} = 0$ whenever $i - j \neq 1$, $i, j \in \mathbb{Z}_p$, by (28) we have

$$\begin{aligned} I_\sigma^p &= \sum_{s\in \mathrm{Perm}(\mathbb{Z}_p)} h_{\sigma_{s(0)}} \circ \cdots \circ h_{\sigma_{s(p-1)}} \\ &= \sum_{s\in \mathrm{Cycle}(\mathbb{Z}_p)} h_{\sigma_{s(0)}} \circ \cdots \circ h_{\sigma_{s(p-1)}} , \end{aligned}$$

which completes the proof of (i).

For the proof of (ii) first observe that, according to the definitions (22) and (26) of I_σ and g_i, respectively, we have the following representation of endomorphisms h_i given by (27):

$$h_{\sigma_{i+1}} = m_{\sigma_i} \circ \pi_{\sigma_i} \circ g_{\sigma_{i+1}} \circ \iota_{\sigma_{i+1}} \circ r_{\sigma_{i+1}}. \tag{29}$$

It is straightforward to see that, for each $i \in I_n$, we have

$$g_{\sigma_i} \circ \iota_{\sigma_i} \circ r_{\sigma_i} \circ m_{\sigma_i} \circ \pi_{\sigma_i} = g_{\sigma_i}. \tag{30}$$

Therefore, using (29), we obtain

$$h_{\sigma_0} \circ \cdots \circ h_{\sigma_{p-1}} = m_{\sigma_{p-1}} \circ \pi_{\sigma_{p-1}} \circ g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}} \circ \iota_{\sigma_{p-1}} \circ r_{\sigma_{p-1}}.$$

This completes the proof. ∎

Proof of Theorem 7.6. According to Theorem 7.5 it suffices to show that I_σ^p is not nilpotent. For contradiction suppose that I_σ^p is nilpotent and consider $k \in \mathbb{N}$ such that $I_\sigma^{kp} = 0$. Note that, by Lemma 7.7 and the fact that $h_{\sigma_j} \circ h_{\sigma_i} = 0$ for $i - j \neq 1$, $i, j \in \mathbb{Z}_p$, it follows that

$$\begin{aligned} I_\sigma^{kp} &= \left(\sum_{s\in \mathrm{Cycle}(\mathbb{Z}_p)} h_{\sigma_{s(0)}} \circ \cdots \circ h_{\sigma_{s(p-1)}} \right)^k \\ &= \sum_{s\in \mathrm{Cycle}(\mathbb{Z}_p)} \left(h_{\sigma_{s(0)}} \circ \cdots \circ h_{\sigma_{s(p-1)}} \right)^k . \end{aligned}$$

Hence, according to definition (27) of h_i, for each $s \in \mathrm{Cycle}(\mathbb{Z}_p)$ we have

$$(h_{\sigma_{s(0)}} \circ \cdots \circ h_{\sigma_{s(p-1)}})^k = 0.$$

In particular, $(h_{\sigma_0} \circ \cdots \circ h_{\sigma_{p-1}})^k = 0$. Consequently, by Lemma 7.7(ii) and (30), we obtain

$$m_{\sigma_{p-1}} \circ \pi_{\sigma_{p-1}} \circ (g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}})^k \circ \iota_{\sigma_{p-1}} \circ r_{\sigma_{p-1}} = 0,$$

which implies $(g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}})^k = 0$, a contradiction. This completes the proof. ∎

8. Determining periodic orbits via Lefschetz-type fixed point theorem. We will continue to deal with determining orbits passing through the disjoint components of an isolating neighborhood in a prescribed fashion. Now we focus our attention on periodic orbits.

Throughout this section we use the notation introduced in the preceding section.

Let $\phi = \{\phi_i\}$ be an endomorphism of degree zero of a graded vector space $V = \{V_i\}$. Recall that ϕ is called a *Leray endomorphism* provided the quotient space $V^0 := V/N(\phi)$, where $N(\phi) := \bigcup \{\phi^{-n}(0) \mid n = 1, 2, \dots\}$, is of a finite type. For such a ϕ we define its trace as a trace of an induced endomorphism $\phi^0 : V^0 \to V^0$, i.e., $\operatorname{tr}(\phi) := \operatorname{tr}(\phi^0)$, and the *(generalized) Lefschetz number*, by

$$\Lambda(\phi) := \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(\phi_i).$$

It is worth mentioning the case of endomorphisms ϕ, ψ of graded vector spaces V and W, respectively, such that $\phi = hg$ and $\psi = gh$ for some morphisms $g : V \to W$ and $h : W \to V$. If one of such endomorphisms is a Leray endomorphism then so is the other, and $\Lambda(\phi^k) = \Lambda(\psi^k)$ for all $k \in \mathbb{N}$ (cf. [13], [26, Proposition 2]). It applies, in particular, if ϕ and ψ are *conjugate*, that is, there exists an isomorphism $g : V \to W$ such that $g\phi = \psi g$.

The following proposition shows that the Lefschetz number of an index map is independent of the choice of a weak index pair.

Proposition 8.1. *Let S be an isolated invariant set for f and let P and R be arbitrary weak index pairs for f and S. Then, for every $k \in \mathbb{N}$, if $\Lambda(I^k_{f_P})$ is well defined, then so is $\Lambda(I^k_{f_R})$ and we have*

$$\Lambda(I^k_{f_P}) = \Lambda(I^k_{f_R}). \tag{31}$$

Proof. By [2, Theorem 6.4] and its proof it follows that there exists a sequence $I_{f_P} = I_1, I_2, \dots, I_k = I_{f_R}$ of endomorphisms, with the property that each two consecutive endomorphisms, I_i and I_{i+1}, are linked in the sense of [26, Proposition 2]. Hence, the assertion follows. ∎

Proposition 8.2. *For any weak index pair Q for $\bar{f}^p$ and S_σ given by (23), if $\Lambda(I_{\bar{f}^p_Q})$ is well defined then so is $\Lambda(I^p_\sigma)$ and we have*

$$\Lambda(I_{\bar{f}^p_Q}) = \Lambda(I^p_\sigma).$$

Proof. By Proposition 6.4, S_σ is an isolated invariant set with respect to both $\bar{f}$ and $\bar{f}^p$. Moreover, according to Proposition 6.5, we can take a pair P^0, which is a weak index pair for each $\bar{f}^k$, $k \in I_p$, and S_σ, and satisfies all the assumptions of Proposition 6.6. Then Proposition 6.6 implies that

$$\Lambda(I_{\bar{f}^p_{P^0}}) = \Lambda(I^p_{\bar{f}_{P^0}}). \tag{32}$$

Since Q is a weak index pair for $\bar{f}^p$ and S_σ, and so is P^0, by Proposition 8.1, we get

$$\Lambda(I_{\bar{f}^p_Q}) = \Lambda(I_{\bar{f}^p_{P^0}}). \tag{33}$$

According to Proposition 7.4 we can take a weak index pair R for $\bar{f}$ and S_σ, such that I_σ^p and $I^p_{\bar{f}_R}$ are conjugate; hence,

$$\Lambda(I^p_{\bar{f}_R}) = \Lambda(I^p_\sigma). \tag{34}$$

Note that both P^0 and R are weak index pairs for $\bar{f}$ and S_σ. Therefore, applying Proposition 8.1 once again, we have

$$\Lambda(I^p_{\bar{f}_{P^0}}) = \Lambda(I^p_{\bar{f}_R}). \tag{35}$$

Now, the assertion follows from (33), (32), (35), and (34). ■

Note that $\bar{f}^p$ maps $X \times \{i\} \subset \bar{X}$ into itself, for any $i \in I_p$. Therefore, the following proposition is straightforward.

Proposition 8.3. *Assume that, for a given $i \in I_p$, $K \times \{i\} \subset \bar{X}$ is an isolated invariant set for $\bar{f}^p$ in its isolating neighborhood $M \times \{i\}$. Then K is an isolated invariant set for f^p, isolated by M.*

Proposition 8.4. *Let $f: \mathbb{R}^d \to \mathbb{R}^d$ be a discrete dynamical system. Set $\overline{\mathbb{R}^d} := \mathbb{R}^d \times I_p$ and consider the dynamical system $\bar{f}$ on $\overline{\mathbb{R}^d}$ given by (17). Assume that $K := \bigcup_{i=0}^{p-1}(K_{\sigma_i} \times \{i\}) \subset \overline{\mathbb{R}^d}$ is an isolated invariant set with respect to $\bar{f}^p$, and $M := \bigcup_{i=0}^{p-1}(M_{\sigma_i} \times \{i\})$ is its isolating neighborhood. Then, there exists a weak index pair Q for $\bar{f}^p$ and K consisting of compact ANR's (for the definition of an ANR we refer to [5]).*

Proof. Fix an arbitrary $i \in I_p$. First note that $K_{\sigma_i} \times \{i\} = \operatorname{Inv}(M_{\sigma_i} \times \{i\}, \bar{f}^p)$, as $\bar{f}^p$ maps $\mathbb{R}^d \times \{i\}$ into itself. As a consequence, $M_{\sigma_i} \times \{i\}$ is an isolating neighborhood of $K_{\sigma_i} \times \{i\}$ with respect to $\bar{f}^p$. By Proposition 8.3, K_{σ_i} is an isolated invariant set with respect to f^p, and M_{σ_i} is its isolating neighborhood. Using [35, Lemma 5.1] we can take a polyhedral index pair Q^{σ_i} for f^p and K_{σ_i}. By [24, Theorem 4.4], Q^{σ_i} is a weak index pair. Then the pair $Q^{\sigma_i} \times \{i\}$ consists of compact ANRs, and constitutes a weak index pair for $\bar{f}^p$ and $K_{\sigma_i} \times \{i\}$. One can verify that the union $Q := \bigcup_{i=0}^{p-1} Q^{\sigma_i} \times \{i\}$ is a weak index pair with respect to $\bar{f}^p$ and K. Moreover, Q_1 and Q_2 are ANRs, as pairwise disjoint unions of ANRs. ■

Theorem 8.5. *Let $f: \mathbb{R}^d \to \mathbb{R}^d$ be a discrete dynamical system. Assume that $N = \bigcup_{i=1}^n N_i$, where N_i are pairwise disjoint compact subsets of N, is an isolating neighborhood with respect to f, and P is a weak index pair for f in N. Let $p \in \mathbb{N}$, let $\sigma := (\sigma_0, \dots, \sigma_{p-1}) \in I_n^{\mathbb{Z}_p}$, and let endomorphism I_σ of $\times_{i=0}^{p-1} H^*(P^{\sigma_i})$ be given by (22). If*

$$\Lambda(I^p_\sigma) \neq 0 \tag{36}$$

then there exists a p-periodic point $x \in N_{\sigma_0}$ for f such that $f^{i+kp}(x) \in N_{\sigma_i}$ for $k \in \mathbb{Z}$.

Proof. Consider the space $\overline{\mathbb{R}^d} := \mathbb{R}^d \times I_p$, and the dynamical system $\bar{f}$ on $\overline{\mathbb{R}^d}$, given by (17). By Proposition 8.2 we infer that S_σ is an isolated invariant set with respect to $\bar{f}^p$. Thus, according to Proposition 8.4, we can take Q, a weak index pair for $\bar{f}^p$ and S_σ, consisting of compact ANRs. Then, by Proposition 8.2, $\Lambda(I_{\bar{f}^p_Q})$ is well defined and we have $\Lambda(I_{\bar{f}^p_Q}) = \Lambda(I^p_\sigma)$ which, along with (36), yields

$$\Lambda(I_{\bar{f}^p_Q}) \neq 0.$$

Note that any weak index pair is a proper pair in the sense of [33, Defnition 4]. Therefore, by [33, Theorem 9], there exists an $\bar{x} \in \operatorname{cl}(Q_1 \setminus Q_2)$ such that $\bar{f}^p_Q(\bar{x}) = \bar{x}$. Without loss of generality we may assume that $\bar{x} = (x, 0) \in N_{\sigma_0} \times \{0\}$. Then, $x \in N_{\sigma_0}$ is a p-periodic point for f. Clearly $\{\bar{f}^k(\bar{x}) \mid k \in \mathbb{Z}\} \subset S_\sigma$; hence, $\{f^k(x) \mid k \in \mathbb{Z}\} \subset \operatorname{Inv}(\bigcup_{i=0}^{p-1} N_{\sigma_i}, f)$. Moreover, definition (17) of f guarantees that the p-periodic trajectory of f through x passes through the components of $\operatorname{Inv}(\bigcup_{i=0}^{p-1} N_{\sigma_i}, f)$ in a proper order. ■

We shall express the Lefschetz number of I^p_σ in terms of the Lefschetz number of a composition of endomorphisms g given by (26). Our goal is to prove the following theorem.

Theorem 8.6. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a discrete dynamical system. Assume that $N = \bigcup_{i=1}^n N_i$, where N_i are pairwise disjoint compact subsets of N, is an isolating neighborhood with respect to f. Let $p \in \mathbb{N}$, let $\sigma := (\sigma_0, \ldots, \sigma_{p-1}) \in I_n^{\mathbb{Z}_p}$, and let P be a weak index pair for f in N. Consider endomorphisms $g_i : H^*(P) \to H^*(P)$ given by (26). If*

$$\Lambda(g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}}) \neq 0 \tag{37}$$

then there exists a p-periodic point $x \in N_{\sigma_0}$ for f such that $f^{i+kp}(x) \in N_{\sigma_i}$ for $k \in \mathbb{Z}$.

For the proof we need an auxiliary lemma.

Lemma 8.7. *Assume I_σ, g_i, and h_i are given by (22), (26), and (27), respectively. Then*
(i) *if $\Lambda(h_0 \circ \cdots \circ h_{p-1})$ is well defined then so is $\Lambda(I^p_\sigma)$, and*

$$\Lambda(I^p_\sigma) = p\Lambda(h_{\sigma_0} \circ \cdots \circ h_{\sigma_{p-1}});$$

(ii) *if $\Lambda(g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}})$ is well defined then so is $\Lambda(I^p_\sigma)$ and we have*

$$\Lambda(I^p_\sigma) = p\Lambda(g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}}).$$

Proof. Note that I^p_σ and $h_{p-1} \circ \cdots \circ h_0$ are endomorphisms of graded modules; however, we consciously skip denoting the dimension in order to simplify the notation. Observe that, by Lemma 7.7(i) and the cyclic property of the trace, in each dimension we have the equality

$$\operatorname{tr}(I^p_\sigma) = p \operatorname{tr}(h_{\sigma_0} \circ \cdots \circ h_{\sigma_{p-1}}). \tag{38}$$

This completes the proof of (i).

For the proof of (ii) it suffices to verify that, in each dimension, we have

$$\operatorname{tr}(I^p_\sigma) = p \operatorname{tr}(g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}}).$$

Using Lemma 7.7(ii), by the cyclic property of the trace, and (30), we can write

$$\begin{aligned}
\operatorname{tr}(h_{\sigma_0} \circ \cdots \circ h_{\sigma_{p-1}}) &= \operatorname{tr}(m_{\sigma_{p-1}} \circ \pi_{\sigma_{p-1}} \circ (g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}}) \circ \iota_{\sigma_{p-1}} \circ r_{\sigma_{p-1}}) \\
&= \operatorname{tr}((g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}}) \circ \iota_{\sigma_{p-1}} \circ r_{\sigma_{p-1}} \circ m_{\sigma_{p-1}} \circ \pi_{\sigma_{p-1}}) \\
&= \operatorname{tr}(g_{\sigma_0} \circ \cdots \circ g_{\sigma_{p-1}}).
\end{aligned}$$

Now, the assertion follows from (i). ■

Proof of Theorem 8.6. The theorem follows from Theorem 8.5 and Lemma 8.7. ■

9. Semiconjugacies to shift dynamics. Given a matrix $A \in \{0,1\}^{I_n \times I_n}$ we say that a partial map $s : \mathbb{Z} \nrightarrow I_n$ is *A-admissible* if $A(s_i, s_{i+1}) = 1$ for any $i, i+1 \in \operatorname{dom} s$.

Assume V is a finite-dimensional graded vector space over the field of rational numbers. Let $V_i \subset V$ for $i \in I_n$ be subspaces of V such that $V = \oplus_{i=1}^n V_i$ is a direct sum decomposition of V and let

$$p_i : V \ni x = (x_1, x_2, \dots x_n) \longmapsto (0, 0, \dots 0, x_i, 0 \dots, 0) \in V$$

denote the canonical projections.

Consider a linear map $L : V \to V$. We define the *transition matrix* of L with respect to the decomposition $V = \oplus_{i=1}^n V_i$ as the matrix $A \in \{0,1\}^{I_n \times I_n}$ such that $A(i,j) = 1$ if and only if $p_j \circ L \circ p_i \neq 0$. We say that L is *Lefschetz-complete* if

$$\Lambda(L \circ p_{s_1} \circ L \circ p_{s_2} \circ \dots \circ L \circ p_{s_k}) \neq 0$$

for any sequence $s : I_k \to I_p$ admissible with respect to the transition matrix of L.

Let $\Sigma_n := \{ s : \mathbb{Z} \to I_n \}$ be the space of bi-infinite sequences of elements in I_n with product topology and for a matrix $A \in \{0,1\}^{I_n \times I_n}$ let Σ_A denote the subspace of A-admissible sequences. It is easy to see that the *shift map* $\sigma : \Sigma_n \to \Sigma_n$ defined by $\sigma(s)_i := s_{i+1}$ is a homeomorphism and $\sigma(\Sigma_A) \subset \Sigma_A$. Hence, σ is a generator of a dynamical system on Σ_A.

Theorem 9.1. *Assume N is an isolating neighborhood with respect to $f : \mathbb{R}^d \to \mathbb{R}^d$, and P is a weak index pair for f in N. Moreover, assume $N = \bigcup_{i=1}^n N_i$, where N_i are pairwise disjoint compact subsets of N, and the index map $I_{f_P} : H^*(P) \to H^*(P)$ is Lefschetz-complete with respect to the decomposition $N = \bigcup_{i=1}^n N_i$. Then there exists a semiconjugacy ρ between $S := \operatorname{Inv}(\bigcup_{i=1}^n N_i, f)$ and the shift dynamics σ on Σ_A, where A is a transition matrix of I_{f_P}. Moreover, for each periodic $s \in \Sigma_A$ there exists a periodic point of f in $\rho^{-1}(s)$.*

Proof. Fix an arbitrary $x \in S$. Since the sets N_i are pairwise disjoint and $S = \operatorname{Inv}(\bigcup_{i=1}^n N_i, f)$, for each $k \in \mathbb{Z}$ there exists a unique $i \in I_n$ with $f^k(x) \in N_i$. By putting $\rho(x)_k := i$ we define a continuous map $\rho : S \to \Sigma_n$. Note that, in fact, ρ maps S into Σ_A, as Σ_A is the subspace of Σ_n of all sequences admissible with respect to the transition matrix of I_{f_P}.

We shall prove that ρ is a surjection onto Σ_A. To this end let $s \in \Sigma_A$ be fixed. For an arbitrary $k \in \mathbb{N}$ let s^k denote the restriction of s to the domain $\{-k, -k+1, \dots, k-1, k\}$. Since I_{f_P} is Lefschetz-complete, we have

$$\Lambda(I_{f_P} \circ p_{s_{-k}} \circ \dots \circ I_{f_P} \circ p_{s_0} \circ \dots \circ I_{f_P} \circ p_{s_k}) \neq 0.$$

By the cyclic property of the trace we obtain

$$\Lambda(I_{f_P} \circ p_{s_k} \circ I_{f_P} \circ p_{s_{-k}} \circ \dots \circ I_{f_P} \circ p_{s_0} \circ \dots \circ I_{f_P} \circ p_{s_{k-1}}) \neq 0,$$

showing that $p_{s_k} \circ I_{f_P} \circ p_{s_{-k}} \neq 0$; hence, (s_{-k}, s_k) is A-admissible. As a consequence, the periodic sequence $\tilde{s}^k : \mathbb{Z} \to I_n$, given by $\tilde{s}^k_m := s^k_{(m+k) \bmod (2k+1) - k}$ for $m \in \mathbb{Z}$, is A-admissible. By Theorem 8.6, there exists $x_k \in S$ such that $\rho(x_k) = \tilde{s}^k$. Since $k \in \mathbb{N}$ was arbitrarily fixed, we have constructed a pair of sequences: $\{\tilde{s}^k\} \in \Sigma_A^{\mathbb{N}}$ convergent to s, and $\{x_k\} \in S^{\mathbb{N}}$, such that $\rho(x_k) = \tilde{s}^k$ for $k \in \mathbb{N}$. By compactness of S, passing to a subsequence if necessary, we may assume that $\{x_k\}$ converges to $x \in S$. Then, by the continuity of ρ we have $\rho(x) = s$.

The commutativity of the diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\rho"'] & S \arrow[d, "\rho"] \\ \Sigma_A \arrow[r, "\sigma"'] & \Sigma_A \end{tikzcd}$$

is easily readable.

The above shows that ρ constitutes a semiconjugacy from f to the shift dynamics σ on Σ_A.

The last statement of the theorem is a direct consequence of Theorem 8.6. ∎

Theorem 9.1 has its counterpart in terms of endomorphisms g given by (26).

Theorem 9.2. *Assume N is an isolating neighborhood with respect to $f : \mathbb{R}^d \to \mathbb{R}^d$, and P is a weak index pair for f in N. Moreover, assume $N = \bigcup_{i=1}^n N_i$, where N_i are pairwise disjoint compact subsets of N, and for each sequence $s : I_k \to I_p$ admissible with respect to the transition matrix A of the index map $I_{f_P} : H^*(P) \to H^*(P)$ the composition $g_{s_1} \circ \cdots \circ g_{s_k}$ is not nilpotent. Then there exists a semiconjugacy ρ between $S := \operatorname{Inv}(\bigcup_{i=1}^n N_i, f)$ and the shift dynamics σ on Σ_A.*

Proof. The proof runs along the lines of the proof of Theorem 9.1. Therefore, the details are left to the reader. However, it is worth mentioning that now the admissibility of the periodic sequence $\tilde{s}^k : \mathbb{Z} \to I_n$ constructed in the proof of Theorem 9.1 follows from the fact that the composition $g_{s_{-k}} \circ \cdots \circ g_{s_0} \circ \cdots \circ g_{s_k}$ is not nilpotent. Moreover, the existence of the corresponding sequence $\{x_k\} \in S^{\mathbb{N}}$ is guaranteed by Theorem 7.6. ∎

10. Proofs of the main theorems.

10.1. Proof of Theorem 1.3. Clearly, F is a cubical map. Its upper semicontinuity follows from [14, Proposition 14.5]. Using elementary collapses (cf. [18]) we verify that F has contractible values.

Using algorithms developed in [36], a formula from [1, Theorem 4.4], and techniques as in [31], we find a cubical isolating block N for F consisting of five pairwise disjoint compact components $N_1, \ldots, N_5$, a cubical weak index pair P in N, and index map F_P (cf. Figure 1.3). Direct computations show that $H^1(P_1, P_2) \cong \mathbb{Z}^5$ and $H^q(P_1, P_2) = 0$ for $q \neq 1$. More precisely, let $\xi^1, \ldots, \xi^5$ be the generators of the cohomology group $H^1(P)$ such that $H^1(P_1^i, P_2^i) = \langle \xi^i \rangle$, where $P^i := P \cap N_i$, for $i = 1, \ldots, 5$. Then, using generators $\xi^1, \ldots, \xi^5$ as a basis, computations based on algorithms of [22] provide the following matrix representation of the index map:

$$I^1_{F_P} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

By Theorem 5.13 we infer that there exists an ε-approximation of F, and each ε-approximation of F shares with F an isolating neighborhood and, up to a conjugacy, an index map.

Property (ii) is a straightforward consequence of Theorem 8.6 under the assumption that the transition matrix A of I_{F_P} is irreducible (cf. [18, Definition 10.22, Proposition 10.25]) and, for any A-admissible periodic sequence σ, condition (37) holds. We verify this assumption by algorithmic computations. Details are presented in [32].

Finally, using the transition matrix A we compute that the topological entropy of f is greater than ln 1.2599.

10.2. Proof of Theorem 1.2. The proof runs along the lines of the proof of Theorem 1.3. Computations result in a cubical isolating block N for F which decomposes into six disjoint compact components $N_1, \ldots, N_6$, and a cubical weak index pair P in N with $H^1(P_1, P_2) \cong \mathbb{Z}^7$ and $H^q(P_1, P_2) = 0$ for $q \neq 1$. Let $P^i := P \cap N_i$ for $i = 1, \ldots, 6$ (cf. Figure 1.2). Then, $H^1(P_1^1, P_2^1)$ has two generators, and each $H^1(P_1^i, P_2^i)$, for $i \neq 1$, has exactly one generator:

$$H^1(P_1^i, P_2^i) = \begin{cases} \langle \xi_1^1, \xi_2^1 \rangle & \text{if } i = 1, \\ \langle \xi^i \rangle & \text{if } i = 2, \ldots, 6. \end{cases}$$

With generators $\xi_1^1, \xi_2^1, \xi^2, \ldots, \xi^6$ as a basis we have the following matrix representation of the index map

$$I^1_{F_P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

The topological entropy of ε-approximation f is greater than ln 1.151.

10.3. Proof of Theorem 1.4. The proof again goes along the lines of the proof of Theorem 1.3. We identify an isolating block $N = N_1 \cup N_2$ with $N_1 \cap N_2 = \emptyset$, and a weak index pair P in N. We find that $H^1(P_1, P_2) \cong \mathbb{Z}^2$ and $H^q(P_1, P_2) = 0$ for $q \neq 1$. More precisely, if ξ^1, ξ^2 are the generators of $H^1(P_1, P_2)$, and let $P^i := P \cap N_i$ then $H^1(P_1^i, P_2^i) = \langle \xi^i \rangle$ for $i = 1, 2$. With the generators as a basis we have the matrix representation of the index map

$$I^1_{F_P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Finally, by Theorem 8.6, we obtain the existence of a 2-periodic point in N.

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